



# ESSENTIALS OF PLANE TRIGONOMETRY AND ANALYTIC GEOMETRY

BY  
ATHERTON H. SPRAGUE  
PROFESSOR OF MATHEMATICS  
AMHERST COLLEGE

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## PREFACE

THE purpose of this book is to present, in a single volume, the essentials of Trigonometry and Analytic Geometry that a student might need in preparing for a study of Calculus, since such preparation is the main objective in many of our freshman mathematics courses. However, despite the connection between Trigonometry and Analytic Geometry, the author believes in maintaining a certain distinction between these subjects, and has brought out that distinction in the arrangement of his material. Hence, the second part of the book—supplemented by the earlier sections on coördinate systems, found in the first part—would be suitable for a separate course in Analytic Geometry, for which a previous knowledge of Trigonometry is assumed.

The oblique triangle is handled by means of the law of sines, the law of cosines, and the tables of squares and square roots. However, the usual law of tangents and the  $r$  formulas are included in an additional chapter, "Supplementary Topics."

There is included in the text abundant problem material on trigonometric identities for the student to solve.

The normal form of the equation of a straight line is derived in as simple a manner as possible, and the perpendicular distance formula is similarly derived from it.

The conics are defined in terms of focus, directrix, and eccentricity; and their equations are derived accordingly.

In the chapter "Transformation of Coördinates" are discussed the general equation of the second degree and the types of conics arising therefrom. An attempt has been made to present rigorously, but without too many details, the material necessary for distinguishing between the types

of conics by means of certain invariants, which enter naturally into the discussion. Although this chapter may be omitted from the course, it is well included if time permits.

Amherst College

ATHERTON H. SPRAGUE

# CONTENTS

## PLANE TRIGONOMETRY

CHAPTER	PAGE
I. LOGARITHMS . . . . .	3
1. Exponents . . . . .	3
2. Definition of a logarithm . . . . .	7
3. Laws of logarithms . . . . .	8
4. Common logarithms . . . . .	10
5. Use of the logarithmic tables . . . . .	12
6. Interpolation . . . . .	13
7. Applications of the laws of logarithms, and a few tricks . . . . .	15
II. THE TRIGONOMETRIC FUNCTIONS . . . . .	21
8. Angles . . . . .	21
9. Trigonometric functions of an angle . . . . .	21
10. Functions of $30^\circ$ , $45^\circ$ , $60^\circ$ . . . . .	23
11. Functions of $(90^\circ - \theta)$ . . . . .	25
12. Tables of trigonometric functions . . . . .	26
III. SOLUTION OF THE RIGHT TRIANGLE . . . . .	29
13. Right triangle . . . . .	29
14. Angles of elevation and depression . . . . .	30
IV. TRIGONOMETRIC FUNCTIONS OF ALL ANGLES . . . . .	35
15. Positive and negative angles . . . . .	35
16. Directed distances . . . . .	35
17. Coördinates . . . . .	36
18. Quadrants . . . . .	37
19. Trigonometric functions of all angles . . . . .	37
20. Functions of $0^\circ$ , $90^\circ$ , $180^\circ$ , $270^\circ$ , $360^\circ$ . . . . .	40
21. Functions of $\theta$ as $\theta$ varies from $0^\circ$ to $360^\circ$ . . . . .	42
22. Functions of $(180 \pm \theta)$ and $(360^\circ \pm \theta)$ . . . . .	45
23. Functions of $(-\theta)$ . . . . .	48

CHAPTER	PAGE
V. THE OBLIQUE TRIANGLE . . . . .	51
24. Law of sines . . . . .	51
25. Applications of the law of sines . . . . .	52
26. Ambiguous case . . . . .	53
27. Law of cosines, and applications . . . . .	57
VI. TRIGONOMETRIC RELATIONS . . . . .	66
28. Fundamental identities . . . . .	66
29. Functions of $(90^\circ + \theta)$ . . . . .	71
30. Principal angle between two lines . . . . .	73
31. Projection . . . . .	73
32. Sine and cosine of the sum of two angles . . . . .	74
33. $\tan(\alpha + \beta)$ . . . . .	76
34. Functions of the difference of two angles . . . . .	78
35. Functions of a double-angle . . . . .	79
36. Functions of a half-angle . . . . .	81
37. Product formulas . . . . .	86
VII. SUPPLEMENTARY TOPICS . . . . .	92
38. Law of tangents . . . . .	92
39. Tangent of a half-angle in terms of the sides of a given triangle . . . . .	94
40. Radius of the inscribed circle . . . . .	98
41. Circular measure of an angle . . . . .	100
42. Summary of trigonometric formulas . . . . .	102
TABLE I: Logarithms to Four Places . . . . .	107
TABLE II: Trigonometric Functions to Four Places . . . . .	111
TABLE III: Squares and Square Roots . . . . .	117

### ANALYTIC GEOMETRY

VIII. COÖRDINATES . . . . .	123
43. Position of a point in a plane . . . . .	123
44. Distance between two points . . . . .	123
45. Mid-point of a line segment . . . . .	125
46. Point that divides a line segment in a given ratio . . . . .	126
47. Slope of a line . . . . .	129
48. Parallel and perpendicular lines . . . . .	130

# CONTENTS

CHAPTER	ix PAGE
VIII. COÖRDINATES ( <i>Con't</i> ).	
49. Angle between two lines . . . . .	131
50. Application of coördinates to plane geometry .	132
IX. LOCUS . . . . .	136
51. Definition and equation of locus . . . . .	136
X. THE STRAIGHT LINE . . . . .	140
52. Equations of lines parallel to the axes . . . . .	140
53. Point-slope form . . . . .	140
54. Slope-intercept form . . . . .	142
55. Two-point form . . . . .	143
56. Intercept form . . . . .	143
57. General form of the equation of a straight line .	144
58. Normal form . . . . .	148
59. Distance from a line to a point . . . . .	152
60. Lines through the point of intersection of two given lines . . . . .	157
XI. THE CIRCLE . . . . .	162
61. Definition and equation of the circle . . . . .	162
62. General form of the equation of the circle. . . .	164
63. Circles through the points of intersection of two given circles; radical axis . . . . .	168
XII. THE PARABOLA . . . . .	172
64. Definition of a conic . . . . .	172
65. Definition and equation of the parabola. . . . .	172
66. Shape of the parabola . . . . .	173
67. Equations of the parabola with vertex not at the origin . . . . .	176
XIII. THE ELLIPSE. . . . .	181
68. Definition and equation of the ellipse. . . . .	181
69. Shape of the ellipse . . . . .	183
70. Second focus and directrix . . . . .	187
71. Equations of the ellipse with center not at the origin . . . . .	188

CHAPTER	PAGE
XIV. THE HYPERBOLA . . . . .	194
72. Definition and equation of the hyperbola. . . . .	194
73. Shape of the hyperbola. . . . .	195
74. Second focus and directrix . . . . .	199
75. Asymptotes. . . . .	199
76. Equations of the hyperbola with center not at the origin. . . . .	202
XV. TRANSFORMATION OF COÖRDINATES . . . . .	209
77. Translation of axes. . . . .	209
78. Rotation of axes. . . . .	213
79. Removal of the $xy$ term. . . . .	215
80. Invariants; classifications of types of conics . . .	219
INDEX. . . . .	223

## PLANE TRIGONOMETRY



## CHAPTER I

### LOGARITHMS

**1. Exponents.** Since a knowledge of the theory of exponents is essential for a clear understanding of logarithms, we shall review briefly that theory. By  $5^3$  we mean

$$5 \times 5 \times 5.$$

By  $a^3$  we mean

$$a \times a \times a.$$

By  $a^m$ , provided  $m$  is a positive integer, we mean

$$a \times a \times a \dots \text{to } m \text{ factors}.$$

We call  $a$  the *base* and  $m$  the *exponent*.

Consider the product of  $5^3 \times 5^4$ .

By this we mean

$$(5 \times 5 \times 5)(5 \times 5 \times 5 \times 5)$$

or

$$5 \times 5 \dots \text{to seven factors}$$

or

$$5^7.$$

Similarly,  $a^m \cdot a^n$  equals

$$(a \times a \dots \text{to } m \text{ factors})(a \times a \dots \text{to } n \text{ factors}).$$

Or:

$$a \times a \dots \text{to } m + n \text{ factors} = a^{m+n}.$$

It is evident that this process gives the law:

*The product of two or more quantities with a common base equals a quantity with the same base and an exponent equal to the sum of the exponents of the various quantities.*

Now consider

$$\frac{5^5}{5^3}$$

By this we mean

$$\frac{\cancel{5} \times \cancel{5} \times \cancel{5} \times 5 \times 5}{\cancel{5} \times \cancel{5} \times \cancel{5}} = 5 \times 5 = 5^2;$$

that is, the number 5 with an exponent equal to the difference of the exponent of the numerator and that of the denominator:

$$5^{5-3}.$$

Or, in general, if  $m$  is greater than  $n$ , then

$$\begin{aligned} \frac{a^m}{a^n} &= \frac{\cancel{a} \times \cancel{a} \dots \text{to } m \text{ factors}}{\cancel{a} \times \cancel{a} \dots \text{to } n \text{ factors}} \\ &= a \times a \dots \text{to } m - n \text{ factors} \\ &= a^{m-n}. \end{aligned}$$

Hence we have the law:

*The quotient of two quantities with a common base equals a quantity with the same base and an exponent equal to the difference of the exponent of the numerator and that of the denominator.*

Now consider

$$\frac{5^3}{5^5}$$

$$\frac{\cancel{5} \times \cancel{5} \times \cancel{5}}{\cancel{5} \times \cancel{5} \times \cancel{5} \times 5 \times 5} = \frac{1}{5^2}$$

However, if the above law is to hold,

$$\frac{1}{5^2} = 5^{3-5} = 5^{-2}.$$

It is quite apparent, therefore, that  $5^{-2}$  does not imply

$$5 \times 5 \dots \text{to } -2 \text{ factors}$$

(which is meaningless), but is a symbol for

$$\frac{1}{5^2}.$$

Similarly, if  $m$  is less than  $n$ , then

$$\frac{a^m}{a^n} = a^{m-n}$$

is a symbol for

$$\frac{1}{a^{n-m}}.$$

Hence we shall define  $a^{-m}$ , when  $m$  is greater than 0 (that is,  $m$  positive) to be equal to

$$\frac{1}{a^m}.$$

In particular, consider

$$\frac{a^m}{a^n}.$$

where  $m$  equals  $n$ ; that is,

$$\frac{a^m}{a^m}.$$

By the above law this equals

$$a^{m-m} = a^0.$$

But we know that

$$\frac{a^m}{a^m} = 1.$$

Therefore, for consistency, we define  $a^0$  to be equal to 1.

Now let us see what we mean by a quantity with a *fractional* exponent. First, what do we mean by the symbol  $\sqrt{3}$ ? We mean that quantity which multiplied by itself gives 3. Let  $\sqrt{3} = 3^x$ , and determine  $x$ . We have

$$3^x \cdot 3^x = 3,$$

or

$$3^{2x} = 3^1.$$

Then

$$2x = 1.$$

Therefore:

$$x = \frac{1}{2}.$$

Hence we define  $3^{\frac{1}{2}}$  to equal

$$\sqrt{3}.$$

In general, we define  $a^{m/n}$  as follows:

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

One fundamental law of exponents remains to be considered. Take  $(5^3)^2$ . By our first law,

$$(5^3)^2 = (5^3)(5^3) = 5^6.$$

Similarly,

$$\begin{aligned}(a^m)^n &= (a^m)(a^m) \dots \text{to } n \text{ factors} \\ &= a^{m+m+\dots} \text{to } n \text{ terms} \\ &= a^{nm} \\ &= a^{mn}.\end{aligned}$$

From these computations we have the law:

*If a quantity with a given base and exponent is raised to a power, the result is a quantity with the same base and an exponent equal to the product of the two exponents.*

The application of all three laws is allowable for fractional exponents, positive and negative, as well as for positive and negative integral exponents.

### Problems

1. Find the value of:  $(243)^{-\frac{2}{5}}$ ;  $\sqrt[12]{84}$ ;  $(-\frac{1}{125})^{\frac{2}{5}}$ ;  $(742)^0$ ;  $(x + 6y - 3)^0$ ;  $1^{-\frac{3}{5}}$ .

2. Express with positive exponents only:

$$\frac{5^{-2}x^{\frac{2}{5}}y^{-\frac{3}{5}}x^{-\frac{1}{5}}}{125^{-\frac{1}{5}}x^2y^{-2}}.$$

3. Express without any denominator:

$$\frac{2x^{-\frac{1}{2}}y^{\frac{3}{5}}x^{\frac{3}{5}}}{3^{-2}y^2x^{-2}}.$$

4. Solve: (a)  $x^{3/4} = 4$ ; (b)  $x^{1/4} = 1$ .

5. Solve:

(a)  $7^x = 49$ .

(c)  $27^x = 3$ .

(e)  $27^x = \frac{1}{9}$ .

(b)  $2^x = 8$ .

(d)  $27^x = 9$ .

(f)  $27^x = \frac{1}{3}$ .

6. Solve:

(a)  $10^x = 1000$ .

(d)  $10^x = 1$ .

(g)  $10^x = .001$ .

(b)  $10^x = 100$ .

(e)  $10^x = .1$ .

(h)  $10^x = .0001$ .

(c)  $10^x = 10$ .

(f)  $10^x = .01$ .

(i)  $10^x = .00001$ .

7. Solve:  $2^{2x} - 6 \cdot 2^x + 8 = 0$ .

8. Solve:  $9 \cdot 3^{2x} - 244 \cdot 3^x + 27 = 0$ . (Answer:  $x = -2$  or 3.)

9. Simplify:

$$\left(\frac{2}{5}\right)^{-3} + 16^{-3/4} + \frac{5}{2^{-2}} + (-2)^{-2}.$$

10. Simplify:  $(3^{n+2} + 3 \cdot 3^n) \div (9 \cdot 3^{n+2})$ .

**2. Definition of a logarithm.** Consider:  $7^2$  equals 49. Observe that 2 is the exponent of the power to which 7 must be raised to give 49. We shall now, by means of a definition, write this expression in another form. We define the *logarithm* of 49 to the base 7 to be equal to 2. Or, expressed in symbols,

$$\log_7 49 = 2.$$

Similarly, since  $8^2$  equals 64, we have

$$\log_8 64 = 2;$$

and since  $2^3$  equals 8, we have

$$\log_2 8 = 3.$$

In general, if  $b^x$  equals  $N$ , we have

$$\log_b N = x;$$

and the general definition:

The *logarithm* of a number  $N$  to the base  $b$  is the exponent

of the power to which the base  $b$  must be raised to give the number  $N$ .

*Example*

Find  $\log_2 64$ .

Let

$$\log_2 64 = x.$$

Then

$$2^x = 64,$$

$$2^x = 2^6,$$

$$x = 6.$$

Therefore:

$$\log_2 64 = 6.$$

**Problems**

Find:

- |                            |                     |                       |                         |
|----------------------------|---------------------|-----------------------|-------------------------|
| 1. $\log_3 27$ .           | 5. $\log_4 64$ .    | 9. $\log_{10} 1000$ . | 13. $\log_{10} .1$ .    |
| 2. $\log_{27} 3$ .         | 6. $\log_9 27$ .    | 10. $\log_{10} 100$ . | 14. $\log_{10} .01$ .   |
| 3. $\log_3 \frac{1}{27}$ . | 7. $\log_{27} 9$ .  | 11. $\log_{10} 10$ .  | 15. $\log_{10} .001$ .  |
| 4. $\log_8 16$ .           | 8. $\log_{125} 5$ . | 12. $\log_{10} 1$ .   | 16. $\log_{10} .0001$ . |

**3. Laws of logarithms.** There are three important laws of logarithms which are immediate consequences of the three laws of exponents and the definition of a logarithm.

The first law is, given the numbers  $M$  and  $N$ :

$$\log_b (MN) = \log_b M + \log_b N.$$

*Proof*

Let

$$\log_b M = x,$$

$$\log_b N = y.$$

Then

$$b^x = M,$$

$$b^y = N.$$

(Since we are interested in the logarithm of the product  $MN$ , we form that product.)

Hence:

$$MN = b^x \cdot b^y;$$

or:

$$MN = b^{x+y}.$$

In logarithmic form,

$$\begin{aligned} \log_b MN &= x + y \\ &= \log_b M + \log_b N. \end{aligned}$$

The second law is:

$$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

*Proof*

As before, let  $\log_b M = x$ ,

$\log_b N = y$ .

Then  $b^x = M$ ,

$b^y = N$ .

Hence:  $\frac{M}{N} = \frac{b^x}{b^y}$ ;

or:  $\frac{M}{N} = b^{x-y}$ .

In logarithmic form,

$$\begin{aligned} \log_b \frac{M}{N} &= x - y \\ &= \log_b M - \log_b N. \end{aligned}$$

The third law is:

$$\log_b M^p = p \log_b M$$

*Proof*

Let  $\log_b M = x$ .

Then  $b^x = M$ .

Hence:  $M^p = b^{px}$

In logarithmic form,

$$\begin{aligned} \log_b M^p &= px \\ &= p \log_b M. \end{aligned}$$

We state these laws as follows:

**Law 1.** *The logarithm of the product of two or more numbers to the same base is the sum of the logarithms of the respective numbers.*

**Law 2.** *The logarithm of the quotient of two numbers to the same base is the difference of the logarithm of the numerator and the logarithm of the denominator.*

**Law 3.** *The logarithm of a number raised to a power is the product of the exponent of the power and the logarithm of the number.*

It is quite apparent that law 3 handles the logarithm of the root of a number.

### Problems

1. Write in expanded form:

$$(a) \log_b \frac{a^2bc}{2\sqrt{d}}; \quad (b) \log_b \frac{3x^3y^{1/2}}{x^2}.$$

2. Write in contracted form:  $2 \log_b x + \frac{1}{2} \log_b y - \log_b xy^2$ .  
 3. Determine which of the following symbols equals  $2 \log_b x$ :

$$(a) \log_b^2 x. \quad (b) \log_b x^2. \quad (c) (\log_b x)^2.$$

**4. Common logarithms.** For numerical computation it is desirable that a universal base be employed, and the most convenient base to use is the base 10. Logarithms with base 10 are called *common*, or *Briggs*, logarithms. (It may be stated, however, that in higher mathematics the base used is the irrational number  $e = 2.71828 \dots$ .) It is understood that in this text the base is 10 unless otherwise stated.

Let us construct a miniature table of logarithms by considering various powers of 10.

$$\begin{array}{lll} 10^4 & = 10,000; & \text{hence, } \log 10,000 = 4 \\ 10^3 & = 1000; & \text{hence, } \log 1000 = 3 \\ 10^2 & = 100; & \text{hence, } \log 100 = 2 \\ 10^1 & = 10; & \text{hence, } \log 10 = 1 \\ 10^0 & = 1; & \text{hence, } \log 1 = 0 \\ 10^{-1} & = .1; & \text{hence, } \log .1 = -1 \end{array}$$

$$10^{-2} = .01; \quad \text{hence, } \log .01 = -2$$

$$10^{-3} = .001; \quad \text{hence, } \log .001 = -3$$

$$10^{-4} = .0001; \quad \text{hence, } \log .0001 = -4$$

Now suppose we wish the logarithm of a number not given here, say 386. Obviously since 386 is between 100 and 1000, the log of 386 is between 2 and 3; that is, 2 plus a fraction less than 1. This fraction, in decimal form, is given in the tables; it is found to be .5866. Hence:

$$\log 386 = 2.5866.$$

Or:

$$10^{2.5866} = 386.$$

Now suppose we wish the log of 38.6. Since this number is between 10 and 100, its log is 1 plus a fraction; from the tables we find the fraction to be the same as that noted above; that is, .5866. Hence:

$$\log 38.6 = 1.5866.$$

From these computations we draw the following conclusions: The logarithm of a number is composed of two parts, an integral part and a fractional or decimal part. We call the integral part, the *characteristic*, and the fractional part, the *mantissa*. For a given number, a shift in the position of the decimal point changes the characteristic but does not change the mantissa.

The characteristic of the log of 386 was found to be 2. It is quite apparent that the characteristic of the log of *every* number between 100 and 1000 is 2; that is, the characteristic of the log of every number with three digits to the left of the decimal point is 2. And it is not difficult to see that the characteristic of the log of a number greater than 1 is always numerically *one less* than the number of digits to the left of the decimal point.

Now let us consider the logs of positive numbers less than 1. Consider log .00386. From our miniature table, since .00386 is between .001 and .01, log .00386 is between  $-3$

and  $-2$ ; that is,  $-3$  plus a positive fraction less than 1, or,  $-3$  plus .5866—which we write as  $\bar{3}.5866$ . Hence:

$$\log .00386 = \bar{3}.5866.$$

The characteristic is  $-3$ , a negative number and numerically *one more* than the number of zeros immediately to the right of the decimal point. In general the characteristic of the log of a positive number less than 1 is negative and numerically one more than the number of zeros immediately to the right of the decimal point. In the latter instance we may always, if we prefer, assume the decimal point to be in a convenient position, compute the characteristic of the log of that number, and then shift the decimal point to its proper position and revise the characteristic by counting.

**5. Use of the logarithmic tables.** Suppose we wish  $\log 726$ . The line from our tables is

N	0	1	2	3	4	5	6	7	8	9
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627

The characteristic is 2. The mantissa is found as follows: Look under  $N$  for the first two digits, 72. Then, since our third digit is 6, our mantissa is on the line of 72 and directly under 6. It is .8609. Hence:

$$\log 726 = 2.8609.$$

Or, suppose we know that the log of a number  $N$  is 1.8591 and we wish to find  $N$ . We look for the number corresponding to the mantissa 8591 and observe it is 723. Then, since the characteristic is 1, the number  $N$  must have two digits to the left of the decimal point. Therefore:

$$N = 72.3.$$

The process of finding a number  $N$  when  $\log N$  is given is obviously the inverse process of finding a log.  $N$  we call the *antilogarithm*. In the process of finding an antilogarithm, the characteristic simply indicates the position

of the decimal point. And it is apparent that, for a positive or zero characteristic, the resulting antilogarithm has *one more* digit to the left of the decimal point than the numerical value of the characteristic; and that, for a negative characteristic, the antilogarithm has *one less* zero immediately to the right of the decimal point than the numerical value of the characteristic. Thus, if  $\log N$  equals 3.8591, then

$$N = .00723.$$

### Problems

1. Find the logarithms of the following:

- |                         |                            |                 |
|-------------------------|----------------------------|-----------------|
| (a) 382.                | (e) $382 \times 10^{-6}$ . | (i) 100.        |
| (b) 3.82.               | (f) 4.61.                  | (j) $10^8$ .    |
| (c) 38,200.             | (g) .000279.               | (k) .0243.      |
| (d) $382 \times 10^6$ . | (h) .00963.                | (l) $10^{-2}$ . |

2. Find the antilogarithms of the following:

- |                      |             |                      |
|----------------------|-------------|----------------------|
| (a) 2.4829.          | (d) 1.7050. | (g) $\bar{3}.9624$ . |
| (b) $\bar{1}.4829$ . | (e) 2.8808. | (h) .8306.           |
| (c) 0.4829.          | (f) 1.9763. | (i) $\bar{2}.0000$ . |

**6. Interpolation.** The logarithms of all numbers of three digits, preceded or followed by any number of zeros, can be found in our tables by the process described in Section 5. We shall now show how, by *interpolation*, we can find logarithms of numbers of four digits.

Consider  $\log 72.63$ . The tables give us  $\log 72.60$  and  $\log 72.70$ . Hence we have:

$$\log 72.60 = 1.8609$$

$$\log 72.63 = (?)$$

$$\log 72.70 = 1.8615$$

We reason as follows: The number 72.63 is three-tenths of the way from 72.60 to 72.70. Hence,  $\log 72.63$  is three-tenths of the way from  $\log 72.60$  to  $\log 72.70$ ; that is, three-tenths of the way from 1.8609 to 1.8615. Subtracting 1.8609 from 1.8615, we find that the *distance*

between the two is 6 units. The desired logarithm is therefore:

$$\begin{aligned} 1.8609 + .3 \times 6 \text{ units} &= 1.8609 + 1.8 \text{ units} \\ &= 1.8609 + 2 \text{ (approx.) units} \\ &= 1.8611. \end{aligned}$$

Hence:

$$\log 72.63 = 1.8611.$$

Similarly,

$$\log 726.3 = 2.8611,$$

and so on.

Find  $\log 384.2$ .

$$\begin{aligned} \log 384.0 &= 2.5843 \\ \log 384.2 &= (?) \\ \log 385.0 &= 2.5855 \end{aligned}$$

$\log 384.2$  is two-tenths of the way from 2.5843 to 2.5855. The distance between them is 12 units.

$$\begin{aligned} .2 \times 12 &= 2.4 \\ &= 2 \text{ (approx.).} \end{aligned}$$

Hence:

$$\log 384.2 = 2.5845.$$

In like manner, interpolation is used in finding antilogarithms of numbers not given in the tables. Consider  $\text{antilog } 2.7463$ . We look in the tables for the two mantissas nearest to 7463, and find 7459 and 7466. Hence, disregarding temporarily the correct position of the decimal point, we have:

$$\begin{aligned} \text{antilog } 7459 &= 5570 \\ \text{antilog } 7463 &= (?) \\ \text{antilog } 7466 &= 5580 \end{aligned}$$

As before,  $\text{antilog } 7463$  must be the same proportion of the way between 5570 and 5580 as 7463 is between 7459

and 7466. We first find what proportion this is. The distance from 7459 to 7466 is 7 units. The distance from 7459 to 7463 is 4 units. Hence, 7463 is four-sevenths of the way from 7459 to 7466. Hence, antilog 7463 is four-sevenths of the way from 5570 to 5580. Therefore:

$$\begin{aligned}\text{antilog } 7463 &= 5570 + \frac{4}{7} \times 10 \text{ units} \\ &= 5570 + \frac{40}{7} \text{ units} \\ &= 5570 + 6 \text{ (approx.) units} \\ &= 5576.\end{aligned}$$

Hence:

$$\text{antilog } 2.7463 = 557.6.$$

### Problems

1. Find the logarithms of:

(a) 3.286.	(d) .0003428.	(g) .01111.
(b) 729.4.	(e) 82.37.	(h) .3263.
(c) 68.43.	(f) 42.94.	(i) .02438.

2. Find the antilogarithms of:

(a) 2.8531.	(d) $\bar{2}.8906$ .	(g) $\bar{2}.2375$ .
(b) 1.9276.	(e) $\bar{1}.8660$ .	(h) .3770.
(c) 4.6081.	(f) 1.0200.	(i) .0964.

7. Applications of the laws of logarithms, and a few tricks.

### Example 1

By logarithms, find:

$$\frac{(24.32)(6.431)}{76.47}.$$

By our first two laws of logarithms, the log of the required number  $N$  may be expressed as follows:

$$\begin{aligned}\log N &= \log 24.32 + \log 6.431 - \log 76.47. \\ \log 24.32 &= 1.3860 \\ \log 6.431 &= 0.8083\end{aligned}$$

Therefore:  $\log \text{numerator} = 2.1943$   
 $\log 76.47 = \underline{1.8835}$   
 Subtracting,  $\log N = .3108$   
 Hence:  $N = 2.045.$

No tricks are necessary in solving the above problem, but suppose the problem reads:

Find:

$$\frac{(24.32)(6.431)}{764.7}.$$

Then  $\log \text{numerator} = 2.1943$   
 $\log 764.7 = \underline{2.8835}$   
 $\log N = (?)$

The solution will involve a negative number. Subtracting correctly, we have:

$$\log N = \bar{1}.3108.$$

However, there is an easier process for solving this problem.

Write 2.1943 as:  $12.1943 - 10.$   
 Then  $\log \text{numerator} = 12.1943 - 10$   
 $\log 764.7 = \underline{2.8835}$   
 Subtracting,  $\log N = 9.3108 - 10$   
 Or, as above,  $\log N = \bar{1}.3108.$

Anticipating negative characteristics and writing them in this manner will be found a very useful practice. From this point in the text we shall write negative characteristics thus.

Suppose we wish

$$\log \frac{N_1}{N_2}$$

when  $\log N_1$  equals  $9.3241 - 10$ , and  $\log N_2$  equals  $9.4762 - 10$ . In this case, the following is the solution:

Write  $\log N_1$  as:  $19.3241 - 20.$

$$\begin{array}{rcl}
 \text{Then} & \log N_1 & = 19.3241 - 20 \\
 & \log N_2 & = \underline{9.4762 - 10} \\
 \text{Subtracting,} & \log \frac{N_1}{N} & = 9.8479 - 10.
 \end{array}$$

*Example 2*

Find  $\sqrt[3]{28.64}$  by logarithms.

By our third law of logarithms,

$$\begin{aligned}
 \log \sqrt[3]{28.64} &= \log (28.64)^{\frac{1}{3}} \\
 &= \frac{1}{3} \log 28.64 \\
 &= \frac{\log 28.64}{3}
 \end{aligned}$$

$$\begin{array}{rcl}
 \text{Then} & \frac{\log 28.64}{3} & = \frac{1.4570}{3} \\
 & & = .4856\frac{2}{3} \\
 & & = .4857.
 \end{array}$$

Therefore:  $\text{antilog } .4857 = 3.060.$

Hence:  $\sqrt[3]{28.64} = 3.060.$

Again no tricks are necessary. But consider the following:

Find  $\sqrt[3]{.0002864}.$

$$\begin{array}{rcl}
 \text{Then} & \frac{\log .0002864}{3} & = \frac{6.4570 - 10}{3} \\
 & & = 2.1523 - 3\frac{1}{3}.
 \end{array}$$

This answer is a bit confusing, however, since we are left with a fractional characteristic. The trouble lies in the fact that 10 is not exactly divisible by 3. Hence, the following is a better solution, since the answer can be handled more readily.\*

\* These two results may be shown to be the same, as follows:

$$\begin{aligned}
 2.1523 - 3\frac{1}{3} &= 2.1523 - 3.3333 \\
 &= 12.1523 - 10 - 3.3333 \\
 &= 8.8190 - 10.
 \end{aligned}$$

Write  $\frac{\log .0002864}{3}$  as:  $\frac{26.4570 - 30}{3}$ .

Then  $\frac{\log .0002864}{3} = 8.8190 - 10$ .

Similarly, consider the following:

Solve:

$$\frac{\log N}{4} = \frac{8.2869 - 10}{4}.$$

Write  $8.2869 - 10$  as:  $38.2869 - 40$ ,  
and so on.

### *Example 3*

Find the amount to which \$100 will grow in 10 years if the interest is compounded semi-annually at 6 per cent.

If the interest were compounded annually, we should have at the end of one year:

$$\$100(1.06);$$

and at the end of 10 years:

$$\$100(1.06)^{10}.$$

If the interest is compounded semi-annually, we shall have at the end of six months:

$$\$100(1.03);$$

and at the end of one year:

$$\$100(1.03)^2;$$

that is,

$$\$100\left(1 + \frac{.06}{2}\right)^2.$$

Hence, at the end of 10 years we shall have:

$$\$100\left[\left(1 + \frac{.06}{2}\right)^2\right]^{10};$$

or:

$$\$100(1.03)^{20}.$$

Now, letting  $x$  equal the amount, we have the following:

$$\begin{aligned}\log x &= \log 100 + 20 \log 1.03 \\ &= 2.0000 + 20(.0128) \\ &= 2.2560.\end{aligned}$$

Therefore:

$$x = 180.3.$$

Our answer is:

$$\$180.30.$$

### Example 4

Solve:  $28.62^x = 684.9$ .

Taking logs of each side, we have the following:

$$\begin{aligned}\log 28.62^x &= \log 684.9 \\ x \log 28.62 &= \log 684.9.\end{aligned}$$

Therefore:

$$\begin{aligned}x &= \frac{\log 684.9}{\log 28.62} \\ &= \frac{2.8356}{1.4567} \\ &= 1.947.\end{aligned}$$

### Problems

1. Find:

$$(a) \frac{(2.382)(69.84)}{(4236)(.02438)}.$$

$$(b) \sqrt[3]{41.72}.$$

$$(c) (3.461)^5.$$

$$(d) \sqrt[3]{\frac{(7.241)(62.86)}{296.3}}.$$

$$(e) \frac{(1.246)^2}{98.77}.$$

$$(f) (328.2)\sqrt[3]{.004691}.$$

$$(g) \frac{(1.286)^2(91.34)^{3/2}}{4.277}$$

$$(h) \left(\frac{637.2}{9885}\right)^5.$$

$$(i) \frac{(72.39)(1.006)^2}{\sqrt[4]{879.3}}.$$

$$(j) \sqrt[7]{679.6}.$$

2. Find the amount to which \$6000 will grow in 5 years if the interest is compounded quarterly at 4 per cent. 7321

3. Solve:  $4.287^x = 52.39$ . 2.73

4. Find:  $(12.43)^{\sqrt{8}}$ . 78.61

5. Prove:  $\log_a N = \log_b N \cdot \log_a b$ .

6. Prove:  $\log_a b \cdot \log_b a = 1$ .

## CHAPTER II

### THE TRIGONOMETRIC FUNCTIONS

**8. Angles.** Suppose a line  $OX$  is revolved about the point  $O$  until it takes the position  $OP$  (Figure 1). An angle  $XOP$  is then generated. We call  $OX$  the *initial side* of the angle, and  $OP$  the *terminal side*. The angle may be denoted by the single letter  $\theta$ . For the moment we shall consider  $\theta$  as acute.

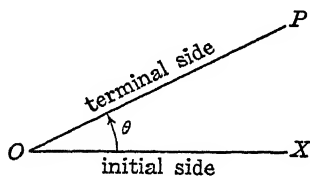


Figure 1.

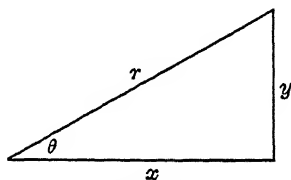


Figure 2

**9. Trigonometric functions of an angle.** Let us take any point on the terminal side and drop a perpendicular to the initial side (extended if necessary). A right triangle is then formed containing  $\theta$  (Figure 2), with legs  $x$  and  $y$ , and hypotenuse  $r$ . Obviously the lengths  $x$ ,  $y$ , and  $r$  are determined by the position of the point chosen, but the *ratios* of any two of the quantities  $x$ ,  $y$ ,  $r$  are unique for a given  $\theta$ . There are six such ratios, called the six *trigonometric functions* of an angle  $\theta$ ; and they are defined as follows:

$$\begin{aligned}\text{sine } \theta &= \sin \theta = \frac{y}{r} = \frac{\text{opposite side}}{\text{hypotenuse}} \\ \text{cosine } \theta &= \cos \theta = \frac{x}{r} = \frac{\text{adjacent side}}{\text{hypotenuse}} \\ \text{tangent } \theta &= \tan \theta = \frac{y}{x} = \frac{\text{opposite side}}{\text{adjacent side}}\end{aligned}$$

$$\text{cosecant } \theta = \csc \theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\text{secant } \theta = \sec \theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\text{cotangent } \theta = \cot \theta = \frac{x}{y} = \frac{\text{adjacent side}}{\text{opposite side}}$$

From these definitions and the law of Pythagoras, all six functions of an acute angle  $\theta$  can be found if any one function is known. For example, suppose we are given

$$\sin \theta = \frac{3}{5}.$$

Since

$$\frac{y}{r} = \frac{3}{5},$$

construct a right triangle with  $y = 3$  and  $r = 5$ , as in Figure 3. Then

$$\begin{aligned} x^2 &= r^2 - y^2 \\ &= 25 - 9 \\ &= 16. \end{aligned}$$

Therefore:

$$x = 4.$$

Then we have:

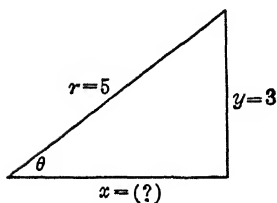


Figure 3.

$$\sin \theta = \frac{3}{5}.$$

$$\cos \theta = \frac{4}{5}.$$

$$\tan \theta = \frac{3}{4}.$$

$$\csc \theta = \frac{5}{3}.$$

$$\sec \theta = \frac{5}{4}.$$

$$\cot \theta = \frac{4}{3}.$$

## Problems

1. Given Figure 2 (see page 21):

- (a) Express  $x$  in terms of  $\cos \theta$ , and  $y$ , or  $r$ .
- (b) Express  $y$  in terms of  $\cot \theta$ , and  $x$ , or  $r$ .
- (c) Express  $r$  in terms of  $\sin \theta$ , and  $x$  or  $y$ .
- (d) Express  $r$  in terms of  $\csc \theta$  and  $x$  or  $y$ .

2. Given a right triangle with hypotenuse  $c$ , opposite side  $b$ , adjacent side  $a$ ; find all functions of  $\theta$  for the following:

- (a)  $a = 3, b = 4, c = 5$ .
- (b)  $a = 5, b = 12, c = 13$ .
- (c)  $a = 1, b = 2, c = \sqrt{5}$ .
- (d)  $a = 2, b = 5, c = \sqrt{29}$ .

3. Given  $\cos \theta = \frac{1}{3}$ ; find  $\sin \theta$ .

4. Given  $\csc \theta = \frac{5}{3}$ ; find  $\tan \theta$ .

5. Given  $\tan \theta = 1$ ; find  $\cot \theta$ .

6. Given  $\cot \theta = \sqrt{3}$ ; find  $\sin \theta$ .

7. Given  $\sin \theta = \frac{1}{\sqrt{2}}$ ; find  $\sec \theta$ .

10. Functions of  $30^\circ, 45^\circ, 60^\circ$ . The functions of  $30^\circ, 45^\circ$ , and  $60^\circ$  can be found from geometric considerations.

We shall find first the functions of  $45^\circ$ . If  $\theta = 45^\circ$ , then, from plane geometry, the right triangle in Figure 2 is isosceles, and  $x = y$ . Hence we have immediately:

$$\tan 45^\circ = 1.$$

In the right triangle in Figure 4,  $x = 1$  and  $y = 1$ ; and we have  $r = \sqrt{2}$ . Consequently we have:

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = 1$$

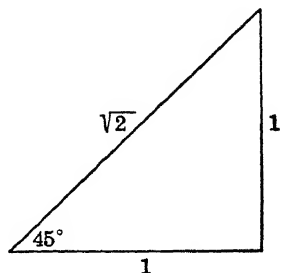


Figure 4.

$$\csc 45^\circ = \sqrt{2}$$

$$\sec 45^\circ = \sqrt{2}$$

$$\cot 45^\circ = 1$$

We shall next find the functions of  $30^\circ$ . If  $\theta = 30^\circ$ , as in Figure 5, then, from plane geometry (Figure 2),  $r = 2y$ . Hence:

$$\frac{y}{r} = \frac{1}{2}$$

and

$$\sin 30^\circ = \frac{1}{2}$$

In the right triangle in Figure 5,  $y = 1$  and  $r = 2$ ; and we have  $x = \sqrt{3}$ . Consequently we have:

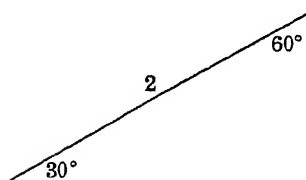


Figure 5.

$$\sin 30^\circ = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\csc 30^\circ = 2$$

$$\sec 30^\circ = \frac{2}{\sqrt{3}}$$

$$\cot 30^\circ = \sqrt{3}$$

It is quite evident from the above explanations that the functions of  $60^\circ$  can be computed from Figure 5; but since the  $60^\circ$  angle may come at the base, we have added Figure 6. Consequently we have:

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\tan 60^\circ = \sqrt{3}$$

$$\csc 60^\circ = \frac{2}{\sqrt{3}}$$

$$\sec 60^\circ = 2$$

$$\cot 60^\circ = \frac{1}{\sqrt{3}}$$

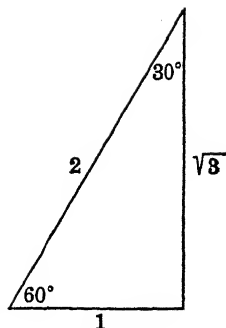


Figure 6.

The above eighteen functions are important because, since the functions are computed geometrically, no tables are necessary for computations with them and, hence, the functions furnish material for a host of problems in many branches of science. We recommend that the student memorize the two basic, simple relations that follow:

$$(1) \sin 30^\circ = \frac{1}{2}$$

$$(2) \tan 45^\circ = 1$$

since from these two relations the remaining sixteen relations can be computed readily.

**11. Functions of  $(90^\circ - \theta)$ .** In a comparison of the functions of  $30^\circ$  and  $60^\circ$ , it is observed that

$$\sin 30^\circ = \cos 60^\circ$$

$$\cos 30^\circ = \sin 60^\circ$$

$$\csc 30^\circ = \sec 60^\circ$$

and so on; that is, the functions of  $30^\circ$  are the corresponding co-functions of  $60^\circ$ . This conclusion suggests a relation between the functions of any acute angle  $\theta$  and the corresponding co-functions of its complement.

We have, in Figure 7,

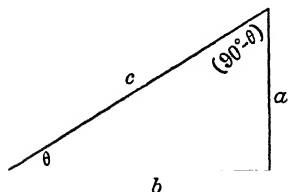


Figure 7.

$$\sin \theta = \frac{a}{c} = \cos (90^\circ - \theta)$$

$$\cos \theta = \frac{b}{c} = \sin (90^\circ - \theta)$$

and so on.

Hence we have:

**Theorem.** Any function of an acute angle  $\theta$  equals the corresponding co-function of  $(90^\circ - \theta)$ , and any function of  $(90^\circ - \theta)$  equals the corresponding co-function of  $\theta$ .

**12. Tables of trigonometric functions.** As illustrated in Section 10, the trigonometric functions of  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$  can be computed geometrically. By more advanced methods the trigonometric functions of other acute angles have been computed and tables have been made, the use of which is similar to that of logarithmic tables. Let us take two typical contiguous lines from our tables.

Degrees	Sine		Tangent		Cotangent		Cosine		
	Value	Log	Value	Log	Value	Log	Value	Log	
<b>38° 00'</b>	.6157	9.7893	.7813	9.8928	1.2799	.1072	.7880	9.8965	<b>52° 00'</b>
<b>10'</b>	.6180	9.7910	.7860	9.8954	1.2723	.1046	.7862	9.8955	<b>50'</b>
<b>39° 00'</b>									<b>51° 00'</b>
	Value	Log	Value	Log	Value	Log	Value	Log	Degrees
	Cosine		Cotangent		Tangent		Sine		

These two lines give us the sine, cosine, tangent, and cotangent of  $38^\circ$ ,  $38^\circ 10'$ ,  $52^\circ$ , and  $51^\circ 50'$ , as well as the logs of these functions, a characteristic 9 standing for 9 - 10, and so on. The tables are so arranged that, for angles from  $0^\circ$  to  $45^\circ$ , we read from the left and from the top, working down. For angles from  $45^\circ$  to  $90^\circ$ , we read from the right and from the bottom, working up. For example, the sine of  $38^\circ$  is .6157, and the sine of  $52^\circ$  is .7880. Observe that the cosine of  $52^\circ$  is .6157, and the cosine of  $38^\circ$  is .7880.  $\sin 38^\circ = \cos 52^\circ$  according to Section 11, since  $38^\circ$  and  $52^\circ$  are complementary angles. Similar results obtain for the tangent and the cotangent.

There are tables of secants and cosecants, but we shall not need them in our work.

The function tables in this book are so arranged, with intervals of 10 minutes, that interpolation is practically like interpolation in logarithms. Suppose, for example, we wish  $\sin 38^\circ 4'$ .

$$\sin 38^\circ = .6157$$

$$\sin 38^\circ 4' = (?)$$

$$\sin 38^\circ 10' = .6180$$

$\sin 38^\circ 4'$  is four-tenths of the way from .6157 to .6180. Therefore:

$$\begin{aligned}\sin 38^\circ 4' &= .6157 + .4(23) \text{ units} \\ &= .6157 + 9 \text{ units} \\ &= .6166.\end{aligned}$$

The tangent is similarly handled.

The cosine of an angle, however, *decreases* as the angle increases, and so is handled a bit differently. Consider  $\cos 38^\circ 4'$ .

$$\cos 38^\circ = .7880$$

$$\cos 38^\circ 4' = (?)$$

$$\cos 38^\circ 10' = .7862$$

$\cos 38^\circ 4'$  is four-tenths of the way from .7880 to .7862. Therefore:

$$\begin{aligned}\cos 38^\circ 4' &= .7880 - .4(18) \text{ units} \\ &= .7880 - 7 \text{ units} \\ &= .7873.\end{aligned}$$

Observe that we have *subtracted* instead of *added*.

The cotangent is handled similarly to the cosine.

### Problems

1. Prove:

$$(a) \sin 60^\circ = 2 \sin 30^\circ \cos 30^\circ.$$

$$(b) \sin 30^\circ = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ.$$

$$(c) \tan 60^\circ = \frac{2 \tan 30^\circ}{1 - \tan^2 30^\circ}.$$

$$(d) \cos 30^\circ = \sqrt{\frac{1 + \cos 60^\circ}{2}}.$$

$$(e) \tan 30^\circ = \frac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}.$$

2. Solve for  $\theta$  in the following equations:

$$(a) \sin \theta = \cos \theta. \quad 45^\circ$$

$$(b) \cos \theta = \sqrt{3} \sin \theta. \quad 30^\circ$$

$$(c) \cos (90^\circ - \theta) = \frac{1}{2}. \quad 30^\circ$$

3. Find:

$$(a) \sin 29^\circ 36'. \quad (c) \sin 62^\circ 13'. \quad (e) \cos 71^\circ 33'.$$

$$(b) \tan 53^\circ 27'. \quad (d) \cos 40^\circ 42'. \quad (f) \cot 42^\circ 26'.$$

4. Find:

$$(a) \log \sin 19^\circ 4'. \quad (c) \log \tan 9^\circ 46'. \quad (e) \log \sin 62^\circ 17'.$$

$$(b) \log \cos 40^\circ 17'. \quad (d) \log \cot 48^\circ 4'. \quad (f) \log \cos 72^\circ 8'.$$

5. Find the angle (a) whose sine is .5883; and (b) whose cosine is .4072.

6. Find the angle (a) whose log tan is  $9.7726 - 10$ ; and (b) whose log cot is 1.6000.

## CHAPTER III

### SOLUTION OF THE RIGHT TRIANGLE

**13. Right triangle.** If we are given a right triangle—and therefore know that one angle is  $90^\circ$ —and we are given, in addition, either of the other angles and any side, or any two sides, the triangle is determined uniquely. It is the purpose of this chapter to show how the trigonometric functions enable one to find the missing parts of a right triangle, when the above information is given. We call this process *solving the right triangle*.

#### Example 1

Given the right triangle  $ABC$  (Figure 8), with  $c = 100$  and  $A = 26^\circ 14'$ ; solve the triangle.

$$\begin{aligned} B &= 90^\circ - 26^\circ 14' \\ &= 63^\circ 46' \end{aligned}$$

$$\frac{a}{c} = \sin A$$

$$\begin{aligned} \therefore a &= c \sin A \\ &= 100 \sin 26^\circ 14' \\ &= 100 \times .4420 \\ &= 44.20 \end{aligned}$$

$$\frac{b}{c} = \sin B$$

$$\begin{aligned} \therefore b &= c \sin B \\ &= 100 \sin 63^\circ 46' \\ &= 100 \times .8970 \\ &= 89.70 \end{aligned}$$

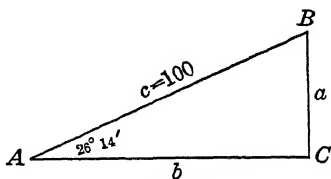


Figure 8.

Or, we might have used the following solution:

$$\frac{b}{c} = \cos A$$

$$\begin{aligned}
 \therefore b &= c \cos A \\
 &= 100 \cos 26^\circ 14' \\
 &= 100 \times .8970 \\
 &= 89.70
 \end{aligned}$$

Logarithms might have been used in this problem; however, having  $c = 100$ , as the multiplier, simplified the problem.

*Example 2*

Given the right triangle  $ABC$ ; find  $a$  when  $b = 382.6$  and  $B = 70^\circ$ .

$$\begin{aligned}
 A &= (90^\circ - B) \\
 &= 20^\circ
 \end{aligned}$$

$$\frac{a}{b} = \tan A$$

$$\therefore a = b \tan A$$

With logarithms,  $\log a = \log b + \log \tan A$   
 $= \log 382.6 + \log \tan 20^\circ$

$$\log 382.6 = 2.5828$$

$$\log \tan 20^\circ = \underline{9.5611 - 10}$$

Adding,  $\log a = 12.1439 - 10$   
 $= 2.1439$

$$\therefore a = 139.3$$

*Example 3*

Given the right triangle  $ABC$ ; find  $A$  when  $c = 389$  and  $a = 202$ .

$$\sin A = \frac{a}{c}$$

$$\begin{aligned}
 \log \sin A &= \log a - \log c \\
 &= \log 202 - \log 389
 \end{aligned}$$

$$\log 202 = 2.3054 - 10$$

$$\log 389 = \underline{2.5899}$$

Subtracting,  $\log \sin A = 9.7155 - 10$

$$\therefore A = 31^\circ 18'$$

**14. Angles of elevation and depression.** The *angle of elevation* of an object above the eye of an observer is defined

as that angle which a line from the observer's eye to the object makes with a horizontal line. Roughly it is the angle through which the observer must *elevate* his eyes to see the object. Thus, in Figure 9, if the observer's eye is at  $E$  and the object is at  $O$ , the angle of elevation is  $\theta$ .

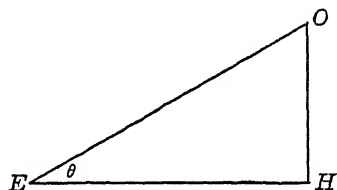


Figure 9.

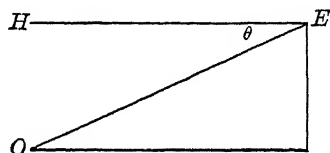


Figure 10.

If, on the other hand, the observer is above the object, the corresponding angle—that is, the angle through which the observer must *depress* his eyes—is called the *angle of depression* of the object. Thus, in Figure 10, if the observer's eye is at  $E$  and the object is at  $O$ , the angle of depression is  $\theta$ .

### Example

A tower stands on the shore of a river 207.2 feet wide (Figure 11). The angle of elevation of the top of the tower from the point on the other shore exactly opposite the tower is  $44^\circ 24'$ . Find the height of the tower.

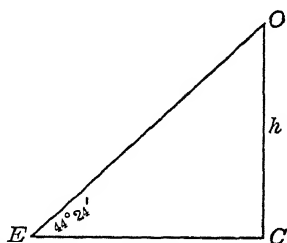


Figure 11.

We are given (assuming the observer's eye on the ground)  $E = 44^\circ 24'$ , and  $EC = 207.2$ . We require  $h$ .

$$\frac{h}{207.2} = \tan 44^\circ 24'$$

Hence  $\log h = \log 207.2 + \log \tan 44^\circ 24'$

$$\log 207.2 = 2.3164$$

$$\log \tan 44^\circ 24' = \underline{9.9909 - 10}$$

Adding,  $\log h = 12.3073 - 10$   
 $= 2.3073$

$$\therefore h = 202.9$$

The height of the tower, therefore, is 202.9 feet.

### Problems

1. Solve the following right triangles, where  $C = 90^\circ$ .

- (a)  $a = 2234$ ,  $A = 36^\circ 19'$ .      (f)  $c = 149.3$ ,  $a = 26.24$ .  
 (b)  $b = 126.3$ ,  $A = 58^\circ 44'$ .      (g)  $c = 60.23$ ,  $B = 68^\circ 43'$ .  
 (c)  $a = 1406$ ,  $b = 2173$ .      (h)  $c = 3204$ ,  $b = 2062$ .  
 (d)  $a = 72.09$ ,  $B = 24^\circ 33'$ .      (i)  $a = 1.263$ ,  $A = 80^\circ 14'$ .  
 (e)  $c = 2434$ ,  $A = 42^\circ 26'$ .      (j)  $b = 2.304$ ,  $A = 22^\circ 46'$ .

2. A ladder 41.24 feet long is so placed that it will reach a window 34.62 feet high on one side of a street. If it is turned over without moving its foot, the ladder will reach a window 20.28 feet high on the other side of the street. Find the width of the street. 58.32

3. A tower stands on the shore of a river 210.6 feet wide. The angle of elevation of the top of the tower from a point on the other shore directly opposite to the tower is  $40^\circ 52'$ . Find the height of the tower. 182.2

4. A rope is stretched from the top of a building to the ground. The rope makes an angle of  $52^\circ 36'$  with the horizontal, and the building is 70 feet high. Find the length of the rope. 86.13

5. The top of a ladder 60.34 feet long rests against a wall at a point 46.23 feet from the ground. Find the angle the ladder makes with the ground, and the distance of its foot from the wall.

6. From the top of a hill the angles of depression of two successive milestones on a straight, level road leading to the hill are  $5^\circ$  and  $15^\circ$ . Find the height of the hill. (Hint: Let  $x$  = height of the hill, and  $y$  = the distance from the bottom to the nearer milestone; then eliminate  $y$  from the two equations representing the cotangent of  $15^\circ$  and  $5^\circ$ , respectively.) 228.7

7. From a point on the ground the angles of elevation of the bottom and the top of a tower on a building are  $64^\circ 17'$  and  $68^\circ 43'$ . If the tower is 200 feet high, find the height of the building. 847

8. Two flag poles are known to be 60 and 40 feet high, respectively. A person moves about until he finds a position such that the tip of the nearer pole just hides that of the farther. At this point the angle of elevation of the top of the nearer pole is found

to be  $35^{\circ} 10'$ . Find the distance between the poles, and the distance from the observer to the nearer pole.  $28.39, 56.78$

9. A tin roof rises  $4\frac{1}{2}$  inches to the horizontal foot. Find the angle the roof makes with the horizontal.  $20^{\circ} 33'$

10. From the top of a mountain, 2800 feet above a hut in the valley, the angle of depression of the hut is found to be  $38^{\circ}$ . Find the straight-line distance from the top of the mountain to the hut.  $4548$

11. Find the height of a tree which casts a shadow of 80 feet when the angle of elevation of the sun is  $40^{\circ}$ .  $67.73$

12. From a ship's masthead 160 feet high, the angle of depression of a boat is  $30^{\circ}$ . Find the distance from the boat to the ship.  $277.1$

13. From the top of a cliff 150 feet high, the angles of depression of two boats at sea, each due south of the observer, are  $32^{\circ}$  and  $20^{\circ}$ , respectively. Find the distance between the boats.  $172$

14. A circle is inscribed in an equilateral triangle of perimeter 90 inches. Find the diameter of the circle.  $17.32$

15. Find the length of a chord which subtends a central angle of  $64^{\circ}$  in a circle whose radius is 10 feet.  $10.59$

16. From the foot of a post 30 feet high, the angle of elevation of the top of a steeple is  $64^{\circ}$ ; and from the top of the post, the angle of depression of the base of the steeple is  $50^{\circ}$ . Find the height of the steeple.  $51.6$

17. From one end of a bridge the angle of depression of an object 200 feet downstream from the bridge and at the water line is  $23^{\circ}$ . From the same point the angle of depression of an object at the water line exactly under the opposite end of the bridge is  $16^{\circ}$ . Find the length of the bridge, and its height above the river.  $296, 84.9$

18. From a point directly north of an inaccessible peak, the angle of elevation of the top is found to be  $28^{\circ}$ . From another point on the same level and directly west of the peak, the angle of elevation of the top is  $40^{\circ}$ . Find the height of the peak if the distance between the two points of observation is 6000 feet. .

19. A tower stands on the bank of a river. The angle of elevation of the top of the tower from a point directly opposite on the other bank is  $55^{\circ}$ . From another point 100 feet beyond this point, the angle of elevation of the top of the tower is  $28^{\circ}$ . Find the height of the tower, and the width of the river.

**20.** A tree stands upon the same horizontal plane as a house which is 60 feet high. The angles of elevation and depression of the top and the base of the tree from the top of the house are  $40^\circ$  and  $35^\circ$ , respectively. Find the height of the tree. 131.9

**21.** The slope of a mountain 4000 feet high makes on one side an angle of  $10^\circ$  with the horizontal, and on the other side an angle of  $12^\circ$ . A man can walk up the steeper slope at the rate of 2 miles per hour, and up the easier slope at 3 miles per hour. Find the route by which he will reach the summit sooner. How many minutes sooner will he arrive?

## CHAPTER IV

### TRIGONOMETRIC FUNCTIONS OF ALL ANGLES

**15. Positive and negative angles.** In Sections 8 and 9, we discussed *acute* angles and defined the six trigonometric functions of acute angles. In this chapter we shall be concerned with angles of any magnitude, positive or negative, and the trigonometric functions of these angles. We first distinguish between positive and negative angles.

If an angle  $\theta$  is generated by a *counter-clockwise* rotation, as in Figure 12, we call angle  $\theta$  a *positive* angle.

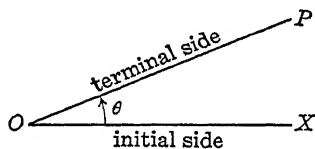


Figure 12.

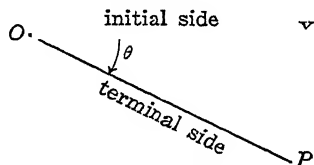


Figure 13.

On the other hand, if an angle  $\theta$  is generated by a *clockwise* rotation, as in Figure 13, we call angle  $\theta$  a *negative* angle.

By an angle of  $390^\circ$ , then, we mean an angle generated by  $OX$  revolving about  $O$  in a counter-clockwise direction, making one complete revolution of  $360^\circ$  and moving  $30^\circ$  in addition, as indicated in Figure 14.

There is a similar understanding for negative angles greater numerically than  $360^\circ$ , the rotation being clockwise.

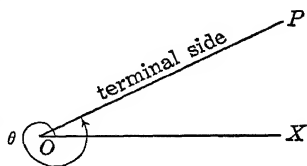


Figure 14.

**16. Directed distances.** Just as in the previous section we made a distinction between positive and negative angles, so now we make a distinction between positive and negative

distances. In general we call a line segment *positive* if it is generated by a point moving from left to right, as  $AB$  in Figure 15. We indicate such a line segment by  $AB$ .



Figure 15.

If the line segment is generated by a point moving from right to left, we call the line segment *negative*, and indicate it by  $BA$ . (Observe that  $AB$  means the segment generated from  $A$  to  $B$ , and that  $BA$  means from  $B$  to  $A$ .) Thus, in Figure 15,

$$BA = -AB.$$

But it is desirable often to distinguish between positive and negative distances when the direction is neither from left to right nor from right to left. In that case, we choose a given direction as positive, and consider the direction directly opposite as negative. Thus,



Figure 16.

in Figure 16, if the direction of the arrow indicates the positive direction, we have:

$$BA = -AB,$$

or:

$$AB = -BA.$$

It is easy to prove that, if  $A$ ,  $B$ , and  $C$  are three points on a line, then, regardless of the positions of  $A$ ,  $B$ , and  $C$ ,

$$AB + BC = AC.$$

**17. Coördinates.** Let  $X'X$  and  $Y'Y$  be two perpendicular lines intersecting at  $O$ . Let  $P$  be any point in the plane of these lines. For purposes of illustration, the point  $P$  is taken as in Figure 17. Drop  $AP$  perpendicular to  $OX$ , and draw  $OP$ . The length  $OA$  is denoted by  $x$ , and is called the  $x$ , or *abscissa*, of  $P$ . The length  $AP$  is denoted

by  $y$ , and is called the  $y$ , or *ordinate*, of  $P$ . The  $x$  and  $y$  taken together, thus  $(x, y)$ , are called the *coördinates* of  $P$ .  $X'X$  and  $Y'Y$  are called the *axes* of the coördinates.  $O$  is called the *origin*, and  $r$  the *radius vector*, of  $P$ . If  $x$  is measured from left to right, we call it positive; if from right to left, negative. If  $y$  is measured upwards, it is positive; if downwards, negative. For convenience we assume  $r$  always to be positive.

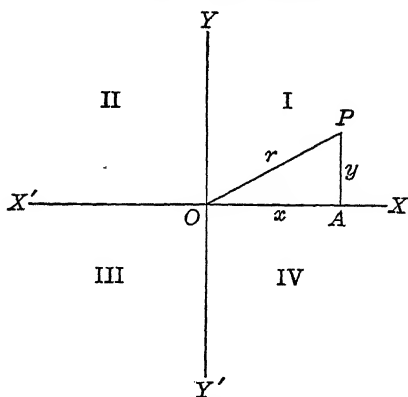


Figure 17.

**18. Quadrants.** The axes divide the plane into four parts, called *quadrants*, numbered as in Figure 17. It is quite apparent then that the following arrangement holds for the signs of the  $x$  and  $y$  of a point in the quadrants indicated.

$P$	$x$	$y$
I	+	+
II	—	+
III	—	—
IV	+	—

Thus the point  $(-2, 3)$  lies in the second quadrant.

**19. Trigonometric functions of all angles.** We shall define the trigonometric functions of angles greater numerically than  $90^\circ$  by a process exactly like the process used in arriving at the definitions of the functions of acute angles. Take the vertex of a given angle at  $O$  (Figure 18), and the initial side along the  $x$ -axis. Take any point  $P$  on the

terminal side, and drop a perpendicular to the initial side, extended if necessary. This forms a right triangle, called the *triangle of reference*—which, it should be noted, however, does not contain angle  $\theta$  unless it is acute.

We define the trigonometric functions of  $\theta$  as follows:

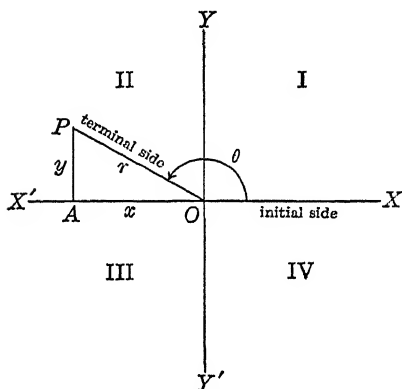


Figure 18.

$$\begin{aligned}\sin \theta &= \frac{y}{r} = \frac{\text{ordinate}}{\text{radius vector}} \\ \cos \theta &= \frac{x}{r} = \frac{\text{abscissa}}{\text{radius vector}} \\ \tan \theta &= \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}} \\ \csc \theta &= \frac{r}{y} = \frac{\text{radius vector}}{\text{ordinate}} \\ \sec \theta &= \frac{r}{x} = \frac{\text{radius vector}}{\text{abscissa}} \\ \cot \theta &= \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}\end{aligned}$$

Observe that, when  $\theta$  is acute and therefore in the first quadrant, the above definitions reduce exactly to those given in Section 9.

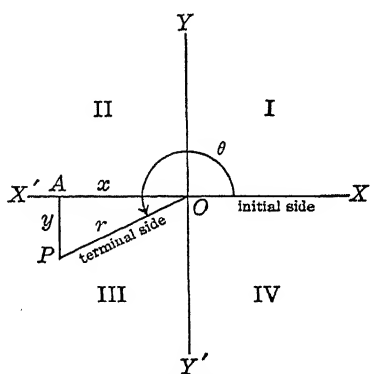


Figure 19a.

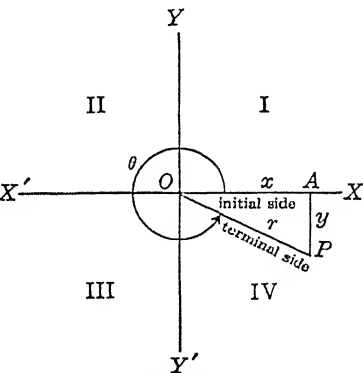


Figure 19b.

Figure 19a and Figure 19b illustrate the situation when  $\theta$  is positive and in the third and fourth quadrants, respectively.

It is quite apparent that, except in the first quadrant where  $x$ ,  $y$ , and  $r$  are all positive quantities, some of the above functions may at times be negative. The following table gives the arrangement of the signs of the functions of angle  $\theta$  in the quadrants indicated; the signs are derived from the table in Section 18 and the definitions in Section 19.

$\theta$	sin	cos	tan	csc	sec	cot
I	+	+	+	+	+	+
II	+	-	-	+	-	-
III	-	-	+	-	-	+
IV	-	+	-	-	+	-

*Example*

The sine of angle  $\theta$  is  $-\frac{3}{5}$ , and the cosine of  $\theta$  is negative. From Figure 20, find the other five functions.

Since  $\sin \theta$  is negative and  $\cos \theta$  is negative,  $\theta$  lies in the third quadrant.

$$\begin{aligned}\text{Then, since } \sin \theta &= -\frac{3}{5} \\ &= \frac{-3}{5} \\ &= \frac{y}{r},\end{aligned}$$

$$\begin{aligned}\text{we have: } x^2 &= r^2 - y^2 \\ &= 25 - 9 \\ &= 16.\end{aligned}$$

$$\text{Therefore: } x = \pm 4.$$

But, since  $\theta$  lies in the third quadrant,  $x$  must be negative.

$$\text{Therefore: } x = -4.$$

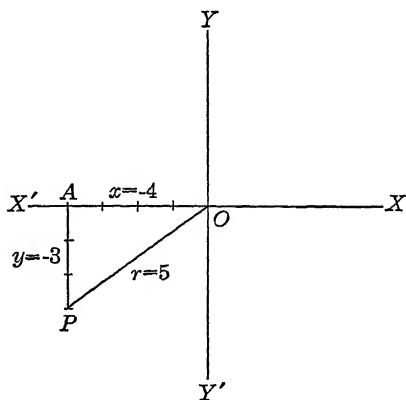


Figure 20.

Hence we have the following:

$$\sin \theta = -\frac{3}{5}$$

$$\cos \theta = -\frac{4}{5}$$

$$\tan \theta = \frac{3}{4}$$

$$\csc \theta = -\frac{5}{3}$$

$$\sec \theta = -\frac{5}{4}$$

$$\cot \theta = \frac{4}{3}$$

### Problems

1. Find in what quadrants the following angles lie:  $346^\circ$ ;  $214^\circ$ ;  $-120^\circ$ ;  $750^\circ$ ;  $-600^\circ$ ;  $-423^\circ$ ;  $542^\circ$ ;  $6000^\circ$ .

In the following problems, consider  $\theta$  positive and between  $0^\circ$  and  $360^\circ$ .

2. Given  $\cos \theta = -\frac{3}{5}$ ,  $\tan \theta$  negative; find all functions of  $\theta$ .
3. Given  $\sec \theta = \frac{5}{4}$ ,  $\sin \theta$  negative; find all functions of  $\theta$ .
4. Given  $\cot \theta = \frac{5}{12}$ ,  $\csc \theta$  negative; find  $\cos \theta$ .
5. Given  $\tan \theta = -\frac{1}{2}$ ,  $\sin \theta$  positive; find  $\cos \theta$ .
6. Given  $\sin \theta = \frac{1}{3}$ ,  $\tan \theta$  positive; find  $\cot \theta$ .
7. Given  $\tan \theta = \frac{2}{5}$ ,  $\theta$  not in the first quadrant; find  $\cos \theta$ .
8. Given  $\sin \theta = -\frac{1}{2}$ ,  $\theta$  not in the fourth quadrant; find  $\sec \theta$ .
9. Given  $\csc \theta = 4$ ; find  $\cos \theta$ .
10. Given  $\tan \theta = 1$ ; find  $\sin \theta$ .

20. Functions of  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$ ,  $270^\circ$ ,  $360^\circ$ . Consider an angle  $\theta$  very close to  $0^\circ$  (Figure 21).

It is quite evident that, when  $\theta = 0^\circ$ , point  $P$  coincides with  $A$ , and hence  $x = r$ , and  $y = 0$ . Therefore:

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{r} = 0$$

$$\cos 0^\circ = \frac{x}{r} = \frac{r}{r} = 1$$

$$\tan 0^\circ = \frac{y}{x} = \frac{0}{x} = 0$$

$$\csc 0^\circ = \frac{r}{y} = \frac{r}{0} = \infty$$

(Infinity\*)

$$\sec 0^\circ = \frac{r}{x} = 1$$

$$\cot 0^\circ = \frac{x}{y} = \frac{x}{0} = \infty$$

Similarly, consider an angle  $\theta$  very close to  $90^\circ$  (Figure 22). When  $\theta = 90^\circ$ , point  $P$  coincides with  $B$ , and we have  $r = y$ , and  $x = 0$ . Therefore:

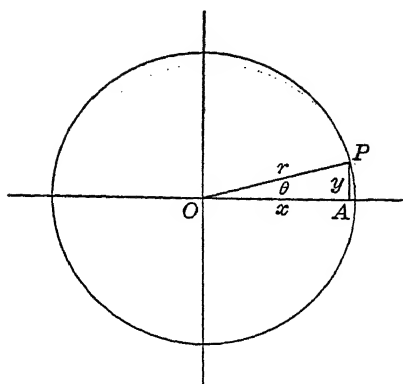


Figure 21.

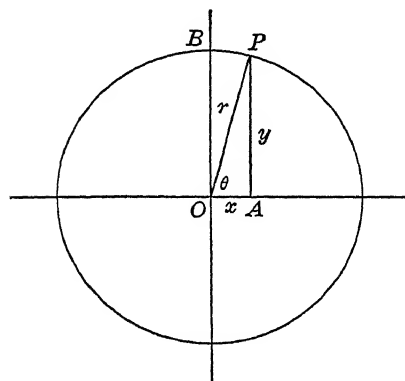


Figure 22.

$$\sin 90^\circ = \frac{y}{r} = 1$$

$$\cos 90^\circ = \frac{x}{r} = \frac{0}{r} = 0$$

$$\tan 90^\circ = \frac{y}{x} = \frac{y}{0} = \infty$$

$$\csc 90^\circ = \frac{r}{y} = 1$$

$$\sec 90^\circ = \frac{r}{x} = \frac{r}{0} = \infty$$

$$\cot 90^\circ = \frac{x}{y} = \frac{0}{y} = 0$$

Similarly, consider an angle  $\theta$  very close to  $180^\circ$  (Figure 23). When  $\theta = 180^\circ$ , point  $P$  coincides with  $A$ . Hence we have  $y = 0$ , and  $x = -r$ ; for  $x$  and  $r$  are equal numerically, but  $x$  is directed to the left and is therefore negative, and  $r$  is always positive. Therefore:

\* By *infinity* we mean that, as  $\theta$  approaches zero,  $y$  approaches zero, and  $\frac{r}{y}$ , that is,  $\csc \theta$ , increases without limit.

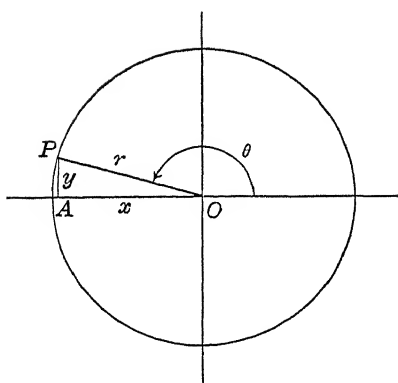


Figure 23.

$$\sin 180^\circ = \frac{y}{r} = \frac{0}{r} = 0$$

$$\cos 180^\circ = \frac{x}{r} = \frac{-r}{r} = -1$$

$$\tan 180^\circ = \frac{y}{x} = \frac{0}{x} = 0$$

$$\csc 180^\circ = \frac{r}{y} = \frac{r}{0} = \infty$$

$$\sec 180^\circ = \frac{r}{x} = \frac{r}{-r} = -1$$

$$\cot 180^\circ = \frac{x}{y} = \frac{x}{0} = \infty$$

Similar results obtain for the functions of  $270^\circ$ , which are tabulated below; and it may readily be seen that the functions of  $360^\circ$  are the same as the corresponding functions of  $0^\circ$ .

$\theta$	$\sin$	$\cos$	$\tan$	$\csc$	$\sec$	$\cot$
$0^\circ$	0	1	0	$\pm \infty$	1	$\pm \infty$
$90^\circ$	1	0	$\pm \infty$	1	$\pm \infty$	0
$180^\circ$	0	-1	0	$\pm \infty$	-1	$\pm \infty$
$270^\circ$	-1	0	$\pm \infty$	-1	$\pm \infty$	0
$360^\circ$	0	1	0	$\pm \infty$	1	$\pm \infty$

**21. Functions of  $\theta$  as  $\theta$  varies from  $0^\circ$  to  $360^\circ$ .** Let us, by means of the above table, study the development of the various functions of an angle as the angle varies from  $0^\circ$  to  $360^\circ$ .

First, consider  $\sin \theta$ .  $\sin \theta$  starts at 0 when  $\theta$  is  $0^\circ$ . Then, as  $\theta$  increases to  $90^\circ$ ,  $\sin \theta$  increases to +1, taking all values between 0 and 1. As  $\theta$  increases from  $90^\circ$  to

$\sin \theta$  decreases from 1 to 0. As  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,  $\sin \theta$  decreases from 0 to  $-1$ . Finally, as  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,  $\sin \theta$  increases from  $-1$  to 0. This process repeats itself as  $\theta$  continues to increase.

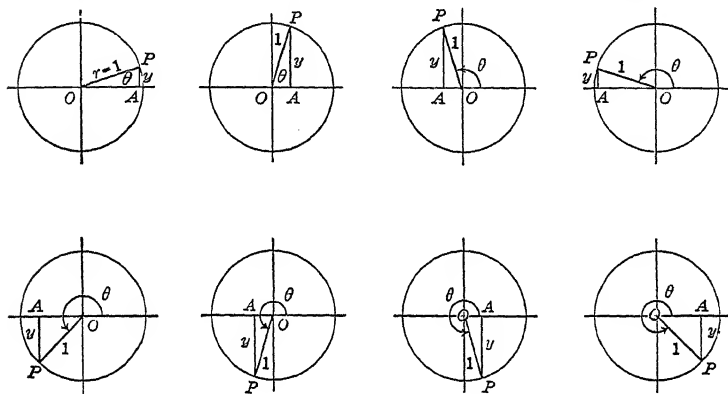


Figure 24.

The process may be illustrated geometrically. Let us consider a circle of radius 1, called a *unit circle* (Figure 24). Then

$$\sin \theta = \frac{y}{r} = \frac{y}{1} = y.$$

$\sin \theta$  may be represented by the line  $AP$ . It is quite apparent from Figure 24 that the development of  $\sin \theta$  as  $\theta$  varies from  $0^\circ$  to  $360^\circ$  is as stated above.

$\cos \theta$  behaves similarly. When  $\theta$  is  $0^\circ$ ,  $\cos \theta$  is 1. As  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ,  $\cos \theta$  decreases from 1 to 0. As  $\theta$  increases from  $90^\circ$  to  $180^\circ$ ,  $\cos \theta$  decreases from 0 to  $-1$ . As  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,  $\cos \theta$  increases from  $-1$  to 0. As  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,  $\cos \theta$  increases from 0 to 1.

When  $\theta$  is  $0^\circ$ ,  $\tan \theta$  is 0. As  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ,  $\tan \theta$  increases from 0 to  $\infty$ . Since the tangent function is negative in the second quadrant and very large numerically for angles slightly larger than  $90^\circ$ , and since it increases

without limit if we approach  $90^\circ$  through such angles, we say  $\tan 90^\circ$  equals  $\pm \infty$ . Hence, as  $\theta$  increases from  $90^\circ$  to  $180^\circ$ ,  $\tan \theta$  increases from  $-\infty$  to 0. As  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,  $\tan \theta$  increases from 0 to  $+\infty$ . As  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,  $\tan \theta$  increases from  $-\infty$  to 0.

When  $\theta$  is  $0^\circ$ ,  $\csc \theta$  is  $\infty$ . As  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ,  $\csc \theta$  decreases from  $\infty$  to 1. As  $\theta$  increases from  $90^\circ$  to  $180^\circ$ ,  $\csc \theta$  increases from 1 to  $\infty$ . As  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,  $\csc \theta$  increases from  $-\infty$  to  $-1$ . As  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,  $\csc \theta$  decreases from  $-1$  to  $-\infty$ .

When  $\theta$  is  $0^\circ$ ,  $\sec \theta$  is 1. As  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ,  $\sec \theta$  increases from 1 to  $\infty$ . As  $\theta$  increases from  $90^\circ$  to  $180^\circ$ ,  $\sec \theta$  increases from  $-\infty$  to  $-1$ . As  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,  $\sec \theta$  decreases from  $-1$  to  $-\infty$ . As  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,  $\sec \theta$  decreases from  $\infty$  to 1.

When  $\theta$  is  $0^\circ$ ,  $\cot \theta$  is  $\infty$ . As  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ,  $\cot \theta$  decreases from  $\infty$  to 0. As  $\theta$  increases from  $90^\circ$  to  $180^\circ$ ,  $\cot \theta$  decreases from 0 to  $-\infty$ . As  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,  $\cot \theta$  decreases from  $+$  to 0. As  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,  $\cot \theta$  decreases from 0 to  $-\infty$ .

The statement  $\tan 90^\circ = \infty$  may be illustrated geometrically by means of a unit circle (Figure 25a).

$$\tan \theta = \frac{AP}{1} = AP$$

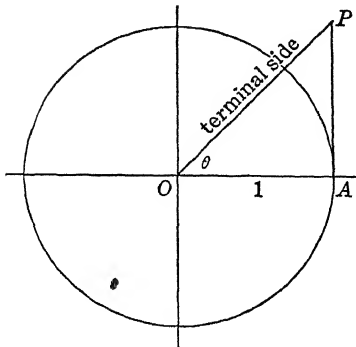


Figure 25a.

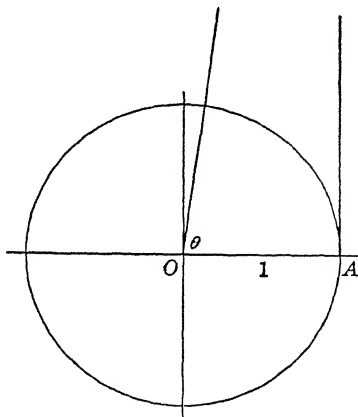


Figure 25b.

$P$  is determined as the intersection point of a tangent to the circle at  $A$  and the terminal side of angle  $\theta$ . It is quite apparent that, as  $\theta$  approaches  $90^\circ$ , the terminal side of  $\theta$  approaches parallelism with the tangent at  $A$ , and  $P$  recedes indefinitely (Figure 25b).

From the above discussion we see that the sine and cosine of an angle are always between  $-1$  and  $+1$ , or equal to  $-1$  or  $+1$ ; that the tangent and cotangent may take any values; that the secant and cosecant are never between  $-1$  and  $+1$ , but may take all other values, including  $-1$  and  $+1$ .

**22. Functions of  $(180^\circ \pm \theta)$  and  $(360^\circ \pm \theta)$ .** In the previous section, we observed that, as  $\theta$  varied from  $0^\circ$  to  $90^\circ$  to  $180^\circ$ ,  $\sin \theta$  varied from  $0$  to  $1$  to  $0$ . It is quite evident, then, if we consider a number between  $0$  and  $1$ —say  $\frac{3}{5}$ —that there is an angle in the first quadrant whose sine is  $\frac{3}{5}$  and that there is, also, an angle in the second quadrant whose sine is  $\frac{3}{5}$ . What is the relation between these two angles? Our answer is: they must be supplementary; that is, the sum of the two angles must equal  $180^\circ$ . We shall prove a general theorem regarding such angles.

**Theorem:**

$$\sin (180^\circ - \theta) = \sin \theta.$$

*Proof*

Given the right triangle  $ABO$  (see Figure 26), with  $\angle XO A = (180^\circ - \theta)$ . Construct  $\angle COX$  equal to  $\theta$ . Take  $OP = r = r'$ . Drop a perpendicular from  $P$  to  $OX$  at  $M$ . Then the right triangles  $MOP$  and  $BOA$  are congruent. Letter as in Figure 26.

Hence we have:

$$\begin{aligned} r &= r', \\ y &= y', \\ x &= -x'. \end{aligned}$$

The sine of  $\angle XOA = (180^\circ - \theta)$  is  $\frac{y'}{r'}$  (see Figure 26). This equals  $\frac{y}{r}$ , since  $y' = y$  and  $r' = r$ . But  $\frac{y}{r}$  (see Figure 26) equals  $\sin \theta$ . Or, restated,

$$\sin (180^\circ - \theta) = \frac{y'}{r'} = \frac{y}{r} = \sin \theta.$$

$$\therefore \sin (180^\circ - \theta) = \sin \theta.$$

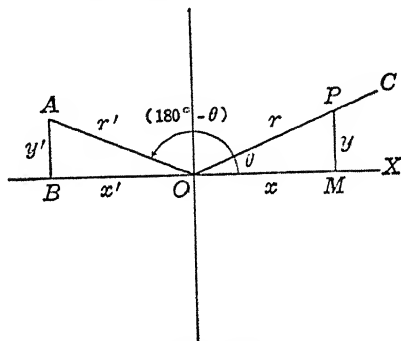


Figure 26.

Similarly,

$$\cos (180^\circ - \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\frac{x}{r} = -\cos \theta.$$

$$\therefore \cos (180^\circ - \theta) = -\cos \theta.$$

Similarly,

$$\tan (180^\circ - \theta) = \frac{y'}{x'} = \frac{y}{-x} = -\frac{y}{x} = -\tan \theta.$$

$$\therefore \tan (180^\circ - \theta) = -\tan \theta.$$

In like fashion,

$$\csc (180^\circ - \theta) = \csc \theta,$$

$$\sec (180^\circ - \theta) = -\sec \theta,$$

$$\cot (180^\circ - \theta) = -\cot \theta.$$

*Example*

Find:  $\sin 150^\circ$ ;  $\cot 135^\circ$ .

$$\sin 150^\circ = \sin (180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}.$$

$$\cot 135^\circ = \cot (180^\circ - 45^\circ) = -\cot 45^\circ = -1.$$

Similar results obtain for the functions of  $(180^\circ + \theta)$ , as indicated in the following text:

*Proof*

In Figure 27, we have:

$$\begin{aligned} r &= r', \\ x &= -x', \\ y &= -y'. \end{aligned}$$

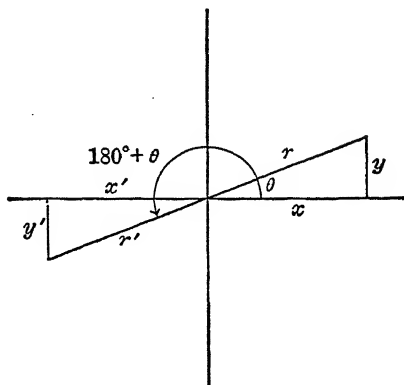


Figure 27.

Hence we have the following:

$$\sin (180^\circ + \theta) = \frac{y'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin \theta$$

$$\cos (180^\circ + \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\frac{x}{r} = -\cos \theta$$

$$\tan (180^\circ + \theta) = \frac{y'}{x'} = \frac{-y}{-x} = \frac{y}{x} = \tan \theta$$

$$\csc (180^\circ + \theta) = -\csc \theta$$

$$\sec (180^\circ + \theta) = -\sec \theta$$

$$\cot (180^\circ + \theta) = \cot \theta$$

Similarly, we have these results:

$$\sin (360^\circ - \theta) = -\sin \theta$$

$$\cos (360^\circ - \theta) = \cos \theta$$

$$\tan (360^\circ - \theta) = -\tan \theta$$

$$\csc (360^\circ - \theta) = -\csc \theta$$

$$\sec (360^\circ - \theta) = \sec \theta$$

$$\cot (360^\circ - \theta) = -\cot \theta$$

From the above discussions we may state the following theorem.

**Theorem.** Any function of  $(180^\circ \pm \theta)$  or  $(360^\circ \pm \theta)$ —in fact, of  $(n180^\circ \pm \theta)$ , where  $n$  is a positive integer—is equal to the same function of  $\theta$ , with the sign depending upon the quadrant in which the angle lies.

Thus:

$$\begin{aligned}\cos 240^\circ &= \cos (180^\circ + 60^\circ) \\ &= -\cos 60^\circ\end{aligned}$$

(since  $240^\circ$  is in the third quadrant and cosine is negative there)

$$= -\frac{1}{2}$$

In the above proofs,  $\theta$  was taken as acute. The theorem holds, however, regardless of the size of  $\theta$ , as does also the information about to be obtained regarding a negative angle.

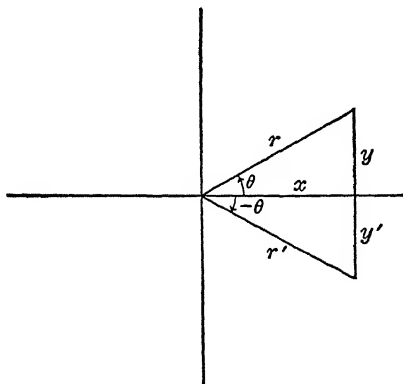


Figure 28.

**23. Functions of  $(-\theta)$ .** What relations hold between the functions of a negative angle and the functions of the corresponding positive angle? Consider Figure 28.

$$r = r'$$

$$x = x$$

$$y = -y'$$

$$\sin (-\theta) = \frac{y'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin \theta$$

$$\cos (-\theta) = \frac{x}{r'} = \frac{x}{r} = +\cos \theta$$

$$\tan (-\theta) = \frac{y'}{x} = \frac{-y}{x} = -\frac{y}{x} = -\tan \theta$$

$$\csc (-\theta) = -\csc \theta$$

$$\sec (-\theta) = \sec \theta$$

$$\cot (-\theta) = -\cot \theta$$

### Problems

#### Example 1

Find:  $\sin -300^\circ$ .

$$\begin{aligned}\sin -300^\circ &= -\sin 300^\circ \\ &= -\sin (360^\circ - 60^\circ) \\ &= -(-\sin 60^\circ) \\ &= \sin 60^\circ \\ &= \frac{\sqrt{3}}{2}.\end{aligned}$$

#### Example 2

Find all angles between  $0^\circ$  and  $360^\circ$  which satisfy the equation:  
 $2 \sin^2 \theta - 3 \sin \theta + 1 = 0$ .

$$2 \sin^2 \theta - 3 \sin \theta + 1 = (2 \sin \theta - 1)(\sin \theta - 1) = 0.$$

Hence:  $\sin \theta = \frac{1}{2},$

or:  $\sin \theta = 1.$

If  $\sin \theta = \frac{1}{2},$

then  $\theta = 30^\circ \text{ or } 150^\circ.$

If  $\sin \theta = 1,$

then  $\theta = 90^\circ.$

Hence:  $\theta = 30^\circ, 150^\circ, \text{ or } 90^\circ.$

1. In Problems (a) to (l), the student is requested not to use the tables. Find:

- (a)  $\sin 210^\circ = -\frac{1}{2}$       (e)  $\sec 225^\circ = -\sqrt{2}$       (i)  $\cos -135^\circ = -\frac{1}{\sqrt{2}}$   
 (b)  $\cot 315^\circ = -1$       (f)  $\cos 120^\circ = -\frac{1}{2}$       (j)  $\cot -300^\circ = \frac{1}{\sqrt{3}}$   
 (c)  $\tan 240^\circ = \sqrt{3}$       (g)  $\tan -330^\circ = -\frac{1}{\sqrt{3}}$       (k)  $\csc -120^\circ = -\frac{2}{\sqrt{3}}$   
 (d)  $\csc 120^\circ = \frac{2}{\sqrt{3}}$       (h)  $\sin -150^\circ = -\frac{1}{2}$       (l)  $\sec -330^\circ = \frac{2}{\sqrt{3}}$

2. Solve the following equations to find, for the unknown letters, all values between  $0^\circ$  and  $360^\circ$ .

$$(a) \cos \theta + \frac{1}{\cos \theta} = \frac{5}{2}.$$

$$(b) \tan^2 \theta + \frac{3}{\tan^2 \theta} - 4 = 0.$$

$$(c) \tan \theta + \frac{1}{\tan \theta} = 2.$$

$$(d) \sin x + \frac{1}{\sin x} = 3.$$

$$(e) \tan x + \sqrt{3} = 0.$$

$$(f) 3 \sin^2 x - 5 \sin x + 2 = 0.$$

$$(g) \sin 2x = 0. \quad 0, 90, 180$$

$$(h) 2 \sin^2 x + \sin x - 1 = 0. \quad 30, 150,$$

$$(i) (\tan^2 \theta - 3)(\csc \theta - 2) = 0. \quad 60, 240$$

$$(j) \sin^2 x = \sin x. \quad 0, 180, 90$$

$$(k) 2 \sin^2 x = \sqrt{3} \sin x. \quad 0, 180, 60, 120.$$

$$(l) \sin^2 \theta - 5 \sin \theta + 6 = 0. \quad -$$

$$(m) 6 \cos^2 x - 5 \cos x + 1 = 0. \quad 60, 300, 70^\circ 32',$$

## CHAPTER V

### THE OBLIQUE TRIANGLE

**24. Law of sines.** In Chapter III we considered the solution of the right triangle. In this chapter we shall be concerned with the solution of the oblique triangle, for which we use two important laws giving relations between the sides and the angles of such a triangle. We proceed to the derivation of the first of these laws, called the *law of sines*:

**Law of Sines.** *The sides of a triangle are proportional to the sines of the angles opposite.*

Consider the triangle  $ABC$  in Figure 29. We wish to prove:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

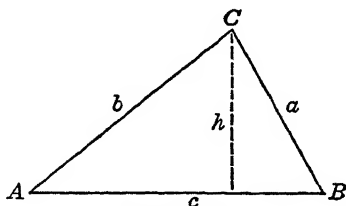


Figure 29.

*Proof*

From  $C$ , drop  $h$  perpendicular to  $AB$ .

Then  $\sin A = \frac{h}{b},$

and  $\sin B = \frac{h}{a}.$

Dividing,  $\frac{\sin A}{\sin B} = \frac{h/b}{h/a} = \frac{a}{b},$

or:  $\frac{a}{\sin A} = \frac{b}{\sin B}.$

Similarly, it may be shown that

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

or

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Hence:

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

In Figure 29, all the angles were taken as acute. The law holds, however, if an angle is obtuse, as  $A$  in Figure 30.

*Proof*

Drop  $h$  perpendicular to  $c$ , extended to  $M$ .

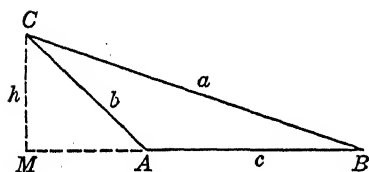


Figure 30.

$$\text{Then} \quad \sin B = \frac{h}{a}$$

$$\text{and} \quad \sin A = \frac{h}{b}$$

by the definition of the sine of an angle in the second quadrant.

The rest of the proof is similar to the preceding proof.

**25. Applications of the law of sines.** By using the law of sines, we are enabled to solve a triangle if we are given any two angles and a side, as in the following:

*Example*

Given  $A = 52^\circ 13'$ ,  $B = 73^\circ 24'$ ,  $c = 6293$ . Solve the triangle.

$$\begin{aligned} C &= 180^\circ - (A + B) \\ &= 179^\circ 60' - 125^\circ 37' \end{aligned}$$

$$\therefore C = 54^\circ 23'$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{c \sin A}{\sin C}$$

$$\begin{aligned}
 \log a &= \log c + \log \sin A - \log \sin C \\
 \log 6293 &= 3.7989 \\
 \log \sin 52^\circ 13' &= \underline{9.8978 - 10} \\
 &13.6967 - 10 \\
 \log \sin 54^\circ 23' &= \underline{9.9101 - 10} \\
 \therefore \log a &= 3.7866 \\
 \therefore a &= 6118
 \end{aligned}$$

$$\begin{aligned}
 \frac{b}{\sin B} &= \frac{c}{\sin C} \\
 \log b &= \log c + \log \sin B - \log \sin C \\
 \log 6293 &= 3.7989 \\
 \log \sin 73^\circ 24' &= \underline{9.9815 - 10} \\
 &13.7804 - 10 \\
 \log \sin 54^\circ 23' &= \underline{9.9101 - 10} \\
 \therefore \log c &= 3.8703 \\
 \therefore c &= 7418
 \end{aligned}$$

Hence:

$$\begin{aligned}
 C &= 54^\circ 23', \\
 a &= 6118, \\
 c &= 7418.
 \end{aligned}$$

### Problems

Solve the following triangles:

1.  $a = 26.32$ ,  $A = 46^\circ 52'$ ,  $B = 64^\circ 43'$ .
2.  $a = 406.2$ ,  $B = 19^\circ 36'$ ,  $C = 80^\circ 52'$ .
3.  $b = 6601$ ,  $A = 50^\circ 32'$ ,  $C = 100^\circ$ .
4.  $c = 32.04$ ,  $A = 25^\circ 42'$ ,  $B = 40^\circ 19'$ .
5.  $c = 530$ ,  $A = 46^\circ 10'$ ,  $B = 63^\circ 50'$ .

**26. Ambiguous case.** The other type of triangle handled by the law of sines is that for which there are given two sides and one angle opposite one of the given sides. This case presents slightly more difficulty, however, because, with the above material, there may be no triangle possible, there may be one and one only, or there may be two triangles possible both of which contain the given material.

This case is therefore known as the *ambiguous case*. Let us consider a concrete illustration:

*Example*

Given  $A = 30^\circ$ ,  $a = 75$ ,  $b = 100$ .

In Figure 31, we construct on  $AX$  at  $A$  an angle of  $30^\circ$ , with sides  $AX$  and  $AM$ . On  $AM$  we lay off 100 units ( $b$ ) from  $A$ , say

With  $C$  as center and ( $a = 75$ ) as radius, we swing an arc.

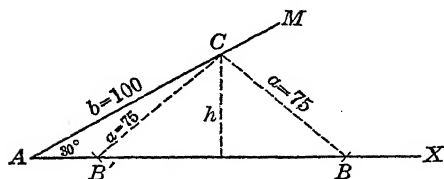


Figure 31.

The number of solutions depends on whether or not the arc cuts  $AX$  and, if so, where. In this particular case, the arc cuts  $AX$  at two points  $B'$  and  $B$ , both to the right of  $A$ . Since both triangles,  $ACB$  and

$ACB'$ , contain the given material, there are consequently two solutions.

Obviously there will always be two solutions when the length of  $a$  is numerically less than  $b$  and greater than the perpendicular  $h$ , dropped from  $C$ . Also, there will never be a solution if  $a$  is less than  $h$ . There will be but one solution, a right triangle, if  $a$  equals  $h$ ; and there will be but one solution, an isosceles triangle, if  $a$  is greater than  $h$  and equal to  $b$ . Finally, there will be but one solution if  $a$  is greater than  $h$  and greater than  $b$ ; for, although the arc will cut  $AX$  at two points, one will be to the left of  $A$  and the triangle thus formed will not include  $A$ .

The matter of finding  $h$  is very simple:

$$\text{Since} \quad \frac{h}{b} = \sin A,$$

$$h = b \sin A.$$

In Figure 31,

$$\begin{aligned} h &= 100 \cdot \frac{1}{2} \\ &= 50. \end{aligned}$$

The above proof and explanation obtain when  $A$  is acute. If  $A$  is obtuse, there will be one and only one solution provided  $a$  is greater than  $b$ , and then only.

We may summarize our findings as follows:

**Case I.** Given  $A$ ,  $a$ , and  $b$ , with  $A$  acute. There will be two solutions if  $b \sin A$  is less than  $a$ , and  $a$  is less than  $b$ . There will be one solution if  $b \sin A$  equals  $a$ , or if  $a$  equals  $b$ , or if  $a$  is greater than  $b$ . There will be no solution if  $a$  is less than  $b \sin A$ .

**Case II.** Given  $A$ ,  $a$ , and  $b$ , with  $A$  obtuse. There will be one solution if  $a$  is greater than  $b$ .

In solving a triangle, the student should realize that the only possible chance for there being two solutions is in the case when  $A$  is acute and  $a$  is less than  $b$ . If such is the case, the student should then find the relation between  $a$  and  $b \sin A$ , and proceed accordingly.

*Example*

Solve the triangle; given  $a = 800$ ,  $b = 1200$ ,  $A = 34^\circ$ . (See Figure 32. This is obviously a chance for two solutions.)

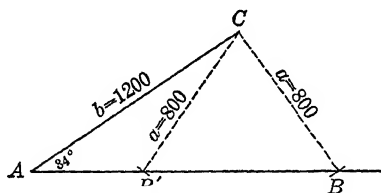


Figure 32.

$$\begin{aligned} \text{Using logs,} \quad \log b &= \log 1200 = 3.0792 \\ \log \sin A &= \log \sin 34^\circ = \underline{9.7476 - 10} \\ \log b \sin A &= 12.8268 - 10 \\ \log a &= \log 800 = 2.9031 \end{aligned}$$

(Since  $\log a$  is greater than  $\log b \sin A$ , there will be two solutions.)

We first solve triangle  $ABC$  in Figure 32.

$$\begin{aligned} \frac{\sin B}{b} &= \frac{\sin A}{a} \\ \sin B &= \frac{b \sin A}{a} \end{aligned}$$

$$\log \sin B = \log b + \log \sin A - \log a = 9.9237 - 10$$

$$\therefore B = 57^\circ 1'$$

$$C = 180^\circ - (A + B) = 180^\circ - 91^\circ 1'$$

$$\therefore C = 88^\circ 59'$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\log c = \log a + \log \sin C - \log \sin A$$

$$\log 800 = 2.9031$$

$$\log \sin 88^\circ 59' = \frac{9.9999 - 10}{12.9030 - 10}$$

$$= 12.9030 - 10$$

$$\log \sin 34^\circ = \frac{9.7476 - 10}{3.1554}$$

$$\log c = 3.1554$$

$$\therefore c = 1430$$

We next solve triangle  $AB'C$ , in Figure 32. Let  $AB' = c'$ , and  $\angle ACB' = C'$ .

$$B = 57^\circ 1'$$

$$\therefore B' = 180^\circ - 57^\circ 1' = 122^\circ 59'$$

$$\therefore C' = \angle ACB' = 180^\circ - (122^\circ 59' + 34^\circ) = 23^\circ 1'$$

$$\frac{c'}{\sin C'} = \frac{a}{\sin A}$$

$$\log c' = \log a + \log \sin C' - \log \sin A$$

$$\log 800 = 2.9031$$

$$\log \sin 23^\circ 1' = \frac{9.5922 - 10}{12.4953 - 10}$$

$$= 12.4953 - 10$$

$$\log \sin 34^\circ = \frac{9.7476 - 10}{2.7477}$$

$$\log c' = 2.7477$$

$$\therefore c' = 559.4$$

Hence, for  $\triangle ABC$ :

$$B = 57^\circ 1',$$

$$C = 88^\circ 59',$$

$$c = 1430;$$

and, for  $\triangle AB'C$ :

$$\begin{aligned} B' &= 122^\circ 59', \\ C' &= 23^\circ 1', \\ c' &= 559.4. \end{aligned}$$

### Problems

1. Find the number of solutions; given:

- (a)  $A = 30^\circ$ ,  $b = 200$ ,  $a = 101$ .
- (b)  $A = 30^\circ$ ,  $b = 400$ ,  $a = 100$ .
- (c)  $A = 30^\circ$ ,  $b = 600$ ,  $a = 300$ .
- (d)  $A = 30^\circ$ ,  $b = 500$ ,  $a = 500$ .
- (e)  $A = 30^\circ$ ,  $b = 500$ ,  $a = 600$ .

2. Find the number of solutions; given:

- (a)  $A = 150^\circ$ ,  $b = 200$ ,  $a = 150$ .
- (b)  $A = 150^\circ$ ,  $b = 200$ ,  $a = 300$ .
- (c)  $A = 150^\circ$ ,  $b = 200$ ,  $a = 200$ .
- (d)  $B = 150^\circ$ ,  $b = 200$ ,  $a = 300$ .
- (e)  $B = 150^\circ$ ,  $c = 400$ ,  $b = 300$ .

3. Solve the following triangles:

- (a)  $A = 59^\circ 26'$ ,  $a = 7072$ ,  $b = 7836$ .
- (b)  $A = 140^\circ 26'$ ,  $a = 40.34$ ,  $b = 30.29$ .
- (c)  $A = 32^\circ 14'$ ,  $a = 464.7$ ,  $b = 600.8$ .
- (d)  $B = 47^\circ 46'$ ,  $b = 3247$ ,  $a = 3015$ .
- (e)  $C = 62^\circ 34'$ ,  $c = 375$ ,  $a = 400$ .
- (f)  $A = 65^\circ 53'$ ,  $a = 20.43$ ,  $b = 30.32$ .

**27. Law of cosines, and applications.** There are two other types of triangles to be considered; that is, triangles with three sides given, or triangles with two sides and the included angle given. These two types are handled by the *law of cosines*. We proceed to its derivation.

*Law of Cosines.* The square of any side of a triangle equals the sum of the squares of the other two sides diminished by twice the product of these sides and the cosine of the included angle.

Given the triangle  $ABC$ , in Figure 33. We wish to prove:

$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}$
---

Let us prove the first relation:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

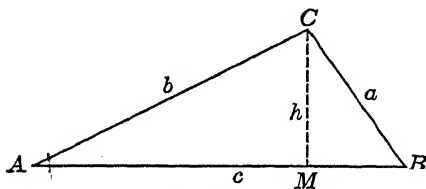


Figure 33.

*Proof*

Drop a perpendicular  $h$  from  $C$  to  $AB$  at  $M$ .

Then

$$h^2 = b^2 - \overline{AM}^2,$$

and

$$h^2 = a^2 - \overline{MB}^2.$$

Equating, we have:

$$a^2 - \overline{MB}^2 = b^2 - \overline{AM}^2,$$

or:

$$a^2 = b^2 + \overline{MB}^2 - \overline{AM}^2$$

Then, since

$$MB = c - AM$$

and

$$\overline{MB}^2 = c^2 - 2c(AM) + \overline{AM}^2,$$

substituting, we have:

$$a^2 = b^2 + c^2 - 2c(AM) + \overline{AM}^2 - \overline{AM}^2.$$

Or:

$$a^2 = b^2 + c^2 - 2c(AM).$$

But

$$\frac{AM}{b} = \cos A,$$

or

$$AM = b(\cos A).$$

Therefore:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

In Figure 33, angle  $A$  was taken as acute. We shall now show that the law holds when  $A$  is obtuse, as in Figure 34.

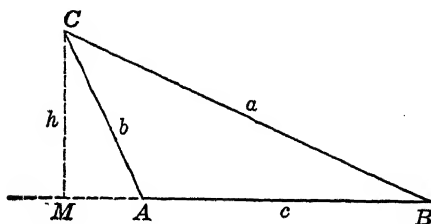


Figure 34.

*Proof*

Drop a perpendicular  $h$  (equal to  $CM$ ) from  $C$  to  $BA$ , extended.

As before,  $h^2 = b^2 - \overline{MA}^2$ ,

and  $h^2 = a^2 - \overline{MB}^2$ .

Then  $a^2 - \overline{MB}^2 = b^2 - \overline{MA}^2$ .

Hence:

$$\begin{aligned} a^2 &= b^2 + \overline{MB}^2 - \overline{MA}^2 \\ &= b^2 + (MA + c)^2 - \overline{MA}^2 \\ &= b^2 + c^2 + 2c(MA) + \overline{MA}^2 - \overline{MA}^2 \\ &= b^2 + c^2 + 2c(MA). \end{aligned}$$

But  $\cos A = \frac{AM}{b}$

from the definition of the cosine of an angle in the second quadrant. However, since

$$AM = -MA,$$

then  $\cos A = -\frac{MA}{b}$ .

Therefore:  $MA = -b \cos A$ .

Hence, substituting,  $a^2 = b^2 + c^2 - 2bc \cos A$ .

The other two relations may be derived in similar fashion

We may apply the law of cosines as in the two examples below.

*Example 1*

Given  $a = 4$ ,  $b = 5$ ,  $c = 6$ ; solve the triangle.

Since  $a^2 = b^2 + c^2 - 2bc \cos A$ ,

then 
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

or 
$$\frac{25 + 36 - 16}{60} = \frac{45}{60} = \frac{3}{4} = .7500$$
  

$$\therefore A = 41^\circ 25'.$$

Similarly, 
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

or 
$$\frac{16 + 36 - 25}{48} = \frac{27}{48} = \frac{9}{16}$$
  

$$\log \cos B = \log 9 - \log 16$$
  

$$\log 9 = 10.9542 - 10$$
  

$$\log 16 = 1.2041$$
  

$$\log \cos B = 9.7501 - 10$$
  

$$\therefore B = 55^\circ 46'.$$

Similarly, 
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

or 
$$\frac{16 + 25 - 36}{40} = \frac{5}{40} = \frac{1}{8} = .1250.$$
  

$$\therefore C = 82^\circ 49'.$$

As a check:  $A + B + C = 180^\circ.$

Of course, as soon as we had found  $A$ , we could have used the law of sines to find  $B$  and subtracted  $(A + B)$  from  $180^\circ$  to find  $C$ . But with convenient numbers, as in this example, it is fully as easy to proceed as above.

### Example 2

Solve the triangle; given  $a = 20$ ,  $b = 25$ ,  $C = 60^\circ$ .

Solving, 
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ &= 400 + 625 - 2 \cdot 500 \cdot \frac{1}{2} \\ &= 1025 - 500 \\ &= 525. \\ \therefore c &= 22.91. \end{aligned}$$

We can now find  $A$  and  $B$  either by the law of cosines or by the law of sines.

It is apparent from the above examples that the law of cosines is not particularly well adapted for the use of logarithms. There are formulas which are better fitted for logarithmic use, but it is the author's feeling that an intelligent use of the tables of squares and square roots combined with the law of cosines is fully as easy and does not involve remembering a set of formulas and their derivations, which are not particularly essential. We shall illustrate in the following:

*Example 3*

Solve the triangle; given  $a = 20.63$ ,  $b = 34.21$ ,  $c = 40.17$ .

We have: 
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

By the table of squares,  $a^2 = 425.6$ ,  
 $b^2 = 1171$ ,  
 $c^2 = 1614$ .

(The interpolation is exactly the same as in logarithms.)

Hence: 
$$\cos A = \frac{2359.4}{2 \times 40.17 \times 34.21}$$

Now we can use logarithms:

$$\begin{aligned} \log \cos A &= \log 2359 - \log 2 - \log 40.17 - \log 34.21 \\ \log 2 &= .3010 \\ \log 40.17 &= 1.6039 \\ \log 34.21 &= \underline{1.5341} \\ \log \text{denominator} &= 3.4390 \\ \log 2359 &= 13.3727 - 10 \\ \log \text{denominator} &= \underline{3.4390} \\ \log \cos A &= 9.9337 - 10 \\ \therefore A &= 30^\circ 51'. \end{aligned}$$

We may proceed similarly to find  $B$  and  $C$ ; or we may use the law of sines. The latter procedure would probably be easier in this example.

### Problems

1. Solve the following triangles:

- (a)  $a = 5, b = 7, c = 10.$
- (b)  $a = 4, b = 6, C = 60^\circ.$
- (c)  $a = 5, b = 8, C = 120^\circ. 11^\circ 36' 21''$
- (d)  $a = 12, b = 20, c = 25. 28^\circ 14' 5''$
- (e)  $a = 19.62, b = 28.43, c = 22.06.$
- (f)  $a = 14.72, c = 25.39, B = 22^\circ 1'$
- (g)  $a = 2032, b = 2491, c = 3824. 2$
- (h)  $a = 1.32, b = 2.63, c = 1.91. 28^\circ 3'$
- (i)  $b = 2.04, c = 3.96, A = 135^\circ 27'.$
- (j)  $a = 423.1, c = 500.2, B = 47^\circ 43'.$

2. Show that the area of a triangle may be written as one-half the product of any two sides and the sine of the included angle.

3. Show that the radius  $R$  of the circle circumscribed about a triangle  $ABC$  is given by

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

4. Show that the area of any quadrilateral equals one-half the product of the diagonals and the sine of one of the included angles.

5. In a parallelogram, the sides are 6 and 15, and the smaller vertex angles are  $50^\circ$ . Find the lengths of the diagonals.  $20.6, 1$

6.  $A$  and  $B$  are points 300 feet apart on the edge of a river, and  $C$  is a point on the opposite side. If the angles  $CAB$  and  $CBA$  are  $70^\circ$  and  $63^\circ$ , respectively, find the width of the river

7. From a mountain top 3000 feet above sea level, two ships are observed, one north and the other northeast. The angles of depression are  $11^\circ$  and  $15^\circ$ . Find the distance between the ships.  $10$

8. A tower stands on one bank of a river. From the opposite bank, the angle of elevation of the tower is  $61^\circ$ ; and from a point 45 feet farther inland, the angle of elevation is  $51^\circ$ . Find the width of the river.  $97.9'$

9. A cliff 400 feet high is seen due south of a boat. The top of the cliff is observed to be at an elevation of  $30^\circ$ . After the boat travels a certain distance southwest, the angle of elevation is found to be  $34^\circ$ . Find how far the boat has gone from the first point of observation.  $123\frac{1}{2}$ ,  $654.6$

10. A vertical tower makes an angle of  $120^\circ$  with the inclined plane on which it stands. At a distance of 80 feet from the base of the tower—measured down the plane—the angle subtended by the tower is  $22^\circ$ . Find the height of the tower.  $48.68$ .

11. Two persons stand facing each other on opposite sides of a pool. The eye of one is 4 feet 8 inches above the water, and that of the other, 5 feet 4 inches. Each observes that the angle of depression of the reflection in the pool of the eye of the other is  $50^\circ$ . Find the width of the pool.  $8.45''$

12. A flag pole stands on a hill which is inclined  $17^\circ$  to the horizontal. From a point 200 feet down the hill, the angle of elevation of the top of the pole is  $25^\circ$ . Find the height of the pole.

13. A tower 100 feet high stands on a cliff beside a river. From a point on the other side of the river and directly across from the tower, the angle of elevation of the top of the tower is  $35^\circ$ , and that of the base of the tower is  $24^\circ$ . Find the width of the river.  $392.2$

14. A ladder leaning against a house makes an angle of  $40^\circ$  with the horizontal. When its foot is moved 10 feet nearer the house, the ladder makes an angle of  $75^\circ$  with the horizontal. Find the length of the ladder.  $19.73$

15. Two forces—one of 10 pounds and the other of 7 pounds—make an angle of  $24^\circ 42'$ . Find the intensity and the direction of their resultant.  $16.62$ ,  $14^\circ 34'$

16. Two men a mile apart on a horizontal road observe a balloon directly over the road. The angles of elevation of the balloon are estimated by the men to be  $62^\circ$  and  $76^\circ$ . Find the height of the balloon above the road.  $1.28$  miles,  $6760$  ft

17. Two points  $A$  and  $B$  are separated by a swamp. To find the length of  $AB$ , a convenient point  $C$  is taken outside the swamp; and  $AC$ ,  $BC$ , and angle  $ACB$  are found as follows:  $AC = 932$  feet,  $BC = 1400$  feet, and  $ACB = 120^\circ$ . Find  $AB$ .  $2$

18. An observer is on a cliff 200 feet above the surface of the sea. A gull is hovering above him, and its reflection in the sea can be seen by the observer. He estimates the angle of eleva-

tion of the gull to be  $30^\circ$ , and the angle of depression of its reflection in the water to be  $55^\circ$ . Find the height of the gull above the sea.

19. An electric sign 40 feet high is put on the top of a building. From a point on the ground, the angles of elevation of the top and the bottom of the sign are  $40^\circ$  and  $32^\circ$ . Find the height of the building. // 6·6

20. A cliff with a lighthouse on its edge is observed from a boat; the angle of elevation of the top of the lighthouse is  $25^\circ$ . After the boat travels 900 feet directly toward the lighthouse, the angles of elevation of the top and the base are found to be  $50^\circ$  and  $40^\circ$ , respectively. Find the height of the lighthouse. 204

21. Two trains start at the same time from the same station upon straight tracks making an angle of  $60^\circ$ . If one train runs 45 miles an hour and the other 55 miles an hour, find how far apart they are at the end of 2 hours. / 73·5

22. From the top of a lighthouse, the angle of depression of a buoy boat at sea is  $50^\circ$ ; and the angle of depression of a second buoy—300 feet farther out to sea but in a straight line with the first buoy—from the top of the lighthouse is  $28^\circ$ . Find the height of the lighthouse. 268

23. A flag pole 50 feet high stands on the top of a tower. From an observer's position near the base of the tower, the angles of elevation of the top and the bottom of the pole are  $36^\circ$  and  $20^\circ$ , respectively. Find the distance from the observer's position to the base of the tower. / 38

24. A lighthouse sighted from a ship bears  $70^\circ$  east of north. After the ship has sailed 6 miles due south, the lighthouse bears  $40^\circ$  east of north. Find the distance of the ship from the lighthouse at each time of observation. 7·7 / , // 28

25. Two trees on a horizontal plane are 60 feet apart. A person standing at the base of one tree observes the angle of elevation of the top of the second. Then, standing at the base of the second tree, he observes that the angle of elevation of one tree is double that of the other. When the observer stands half-way between the trees, the angles of elevation are complementary. Find the height of each tree. 20, 45

26. Two points are in a line, horizontally, with the base of a tower. Let  $\alpha$  be the angle of elevation of the top of the tower from the nearer point, and  $\beta$  the angle of elevation from the far-

ther point. Show that, if  $d$  represents the distance between the points, the height of the tower is

$$\frac{d \sin \alpha \sin \beta}{\sin (\alpha - \beta)}$$

27. A man on a cliff, at a height of 1320 feet, looks out across the ocean. (The radius of the earth is assumed to be 4000 miles.) Find the distance from the man to the horizon seen by him. 44.75

28. Find how high an observer must be above the surface of the ocean to see an object 30 miles distant on the surface. 144

29. From the top of a building at a distance  $d$  from a tower, the angle of elevation of the top of the tower is  $\alpha$ , and the angle of depression of the base is  $\beta$ . Show that the height of the tower is

$$\frac{d \sin (\alpha + \beta)}{\cos \alpha \cos \beta}.$$

30. If  $r$  is the radius of the earth,  $h$  the height of an observer above sea level, and  $\alpha$  the angle of depression of the observer's horizon, show that

$$\tan \alpha = \frac{\sqrt{2rh + h^2}}{r}.$$

31. A balloon is overhead. An observer, due north, estimates the angle of elevation to be  $\alpha$ . Another observer, at a distance  $d$  due west from the first observer, figures his angle of elevation to be  $\beta$ . Show that the height of the balloon above the observers is

$$\frac{d \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}.$$

## CHAPTER VI

### TRIGONOMETRIC RELATIONS

**28. Fundamental identities.** This chapter is concerned with relations of the trigonometric functions of angles of various size and formation. In the present section we shall derive the so-called *fundamental identities*. Although, in Figure 35, the angle under consideration is acute, any angle might have been used.

The following relations are immediate consequences of the definitions of the six trigonometric functions of an angle:

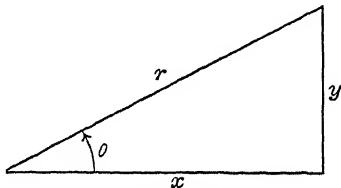


Figure 35.

$$(1) \csc \theta = \frac{1}{\sin \theta}$$

$$(2) \sec \theta = \frac{1}{\cos \theta}$$

$$(3) \cot \theta = \frac{1}{\tan \theta}$$

$$(4) \frac{1}{\csc \theta} = \sin \theta$$

$$(5) \frac{1}{\sec \theta} = \cos \theta$$

$$(6) \frac{1}{\cot \theta} = \tan \theta$$

The first relation is proved as follows:

$$\csc \theta = \frac{r}{y} = \frac{1}{y/r} = \frac{1}{\sin \theta}.$$

The other relations are proved similarly.

We have also:

$$(7) \frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$(8) \frac{\cos \theta}{\sin \theta} = \cot \theta$$

To prove the first, we substitute and have:

$$\frac{\sin \theta}{\cos \theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan \theta.$$

The second is proved in similar fashion.

There remain to be discussed three other important identities:

$$(9) \sin^2 \theta + \cos^2 \theta = 1$$

$$(10) 1 + \tan^2 \theta = \sec^2 \theta$$

$$(11) 1 + \cot^2 \theta = \csc^2 \theta$$

To prove these three relations, we apply the law of Pythagoras to the triangle in Figure 35:

$$y^2 + x^2 = r^2.$$

Dividing both sides of this equation by  $r^2$ , we have:

$$\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = 1,$$

or:

$$\sin^2 \theta + \cos^2 \theta = 1.$$

Similarly, dividing by  $x^2$ , we obtain the second relation; and dividing by  $y^2$ , we obtain the third.

Since these relations are of fundamental importance, the student should memorize all of them.

NOTE: The object in *proving an identity* is to reduce both sides of the given relation to the same quantity. This may be done by working with the left-hand side alone, or with the right-hand side alone, or by working with both sides. In the last instance, we feel that the problem is aesthetically a bit more nicely done if the two sides are not combined;

moreover, the practice of combining the sides frequently leads to errors in computation. Thus, suppose we wish to prove the following:

$$-2 = 2.$$

Squaring, we have

$$4 = 4,$$

which is true. Hence, we reason, the original relation

$$-2 = 2$$

is true; but this conclusion is obviously absurd.

### Example 1

Prove the following identity:

$$\frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} = \sec \theta \csc \theta.$$

Since

$$\cos^2 \theta + \sin^2 \theta = 1,$$

the left-hand side becomes:

$$\frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} = \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta}$$

The right-hand side becomes:

$$\frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta} = \frac{1}{\sin \theta \cos \theta}.$$

### Example 2

Prove:

$$(1 + \cot^2 \theta) \cos^2 \theta = \csc^2 \theta.$$

Since

$$1 + \cot^2 \theta = \csc^2 \theta,$$

the left-hand side becomes:

$$\begin{aligned} \csc^2 \theta \cdot \cos^2 \theta &= \frac{1}{\sin^2 \theta} \cdot \cos^2 \theta \\ &= \frac{\cos^2 \theta}{\sin^2 \theta}, \end{aligned}$$

which the right-hand side also equals.

*Example 3*

Prove:

$$\frac{1 + \sin \theta}{\cos \theta} = \frac{\cos \theta}{1 - \sin \theta}.$$

We know

$$-\sin^2 \theta = \cos^2 \theta.$$

That suggests multiplying both numerator and denominator of the left-hand side by  $(1 - \sin \theta)$ . We have then:

$$\begin{aligned} \frac{1 + \sin \theta}{\cos \theta} &= \frac{1 - \sin^2 \theta}{\cos \theta (1 - \sin \theta)} \\ &= \frac{\cos^2 \theta}{\cos \theta (1 - \sin \theta)} \\ &= \frac{\cos \theta}{1 - \sin \theta}, \end{aligned}$$

which the right-hand side also equals.

It is quite apparent from the above that there is no set rule to follow in proving identities; but, in general, a safe rule is:

*Reduce everything to sines and cosines. Then, wherever necessary, make use of the identity:  $\sin^2 \theta + \cos^2 \theta = 1$ .*

In later sections of the text where we are considering relations involving double-angles, half-angles, and so forth, it will generally be found desirable to reduce our quantities to functions of a *single* angle.

Of course, any time the quantity  $(1 + \tan^2 \theta)$  appears, we may substitute, first,

$$\sec^2 \theta$$

and, then,

$$\frac{1}{\cos^2 \theta}.$$

If we forget that particular identity, the above method of reducing everything to sines and cosines will still hold.

Also, in general, if one side of an identity to be proved is more complicated than the other, it is advisable to reduce the more complicated side first.

### Problems

Prove the following identities:

1.  $\tan \theta + \cot \theta = \sec \theta \csc \theta$ .
2.  $\cos \theta \tan \theta = \sin \theta$ .
3.  $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$ .
4.  $(\sin A - \cos A)^2 = 1 - 2 \sin A \cos A$ .
5.  $(\sin^2 A + \cos^2 A)^2 = 1$ .
6.  $(1 + \sec A)(1 - \cos A) = \tan^2 A \cos A$ .
7.  $\sec \theta - 1 = \sec \theta(1 - \cos \theta)$ .
8.  $\cos \theta + \tan \theta \sin \theta = \sec \theta$ .
9.  $\sin X(1 + \tan X) + \cos X(1 + \cot X) = \sec X + \csc X$ .
10.  $\cos X \csc X \tan X = \sin X \sec X \cot X$ .
11.  $\csc^4 A - \cot^4 A = \csc^2 A + \cot^2 A$ .
12.  $(1 + \tan \theta)(1 + \cot \theta) = (1 + \tan \theta) + (1 + \cot \theta)$ .
13.  $(\cos^2 \theta - 1)(\cot^2 \theta + 1) + 1 = 0$ .
14.  $\sin \theta \cos \theta(\sec \theta + \csc \theta) = \sin \theta + \cos \theta$ .
15.  $(\tan \theta - \sin \theta)^2 + (1 - \cos \theta)^2 = (\sec \theta - 1)^2$ .
16.  $\cot^2 X \cos^2 X = \cot^2 X - \cos^2 X$ .
17.  $(\sin A + \csc A)^2 + (\cos A + \sec A)^2 - \tan^2 A - \cot^2 A = 7$ .
18.  $\sin^4 \theta - \cos^4 \theta = \sin^2 \theta - \cos^2 \theta$ .
19.  $\sin^3 \theta + \cos^3 \theta = (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta)$ .
20.  $\cos^3 \theta - \sin^3 \theta = (\cos \theta - \sin \theta)(1 + \sin \theta \cos \theta)$ .
21.  $1 - \tan^4 B = 2 \sec^2 B - \sec^4 B$ .
22.  $(\sin^2 A - \cos^2 A)^2 = 1 - 4 \cos^2 A + 4 \cos^4 A$ .
23.  $\frac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$ .
24.  $\frac{1 - \sin A}{1 + \sin A} (\sec A - \tan A)^2$ .
25.  $\frac{1}{1 + \tan^2 A} + \frac{1}{1 + \cot^2 A} = 1$ .
26.  $\frac{\sec \theta - \csc \theta}{\sec \theta + \csc \theta} - \frac{\tan \theta - 1}{\tan \theta}$

27.  $\frac{1 + 2 \cos \theta}{\sin \theta} = \csc \theta + 2 \cot \theta.$
28.  $\frac{\tan A - 1}{\tan A + 1} = \frac{1 - \cot A}{1 + \cot A}.$
29.  $\frac{1}{1 + \cos^2 A} = \frac{\sec^2 A}{\tan^2 A + 2}.$
30.  $\frac{1}{\tan A + \cot A} = \sin A \cos A.$
31.  $\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = 2 \csc A.$
32.  $\tan A - \sin A = \frac{\sec A \sin^3 A}{1 + \cos A}.$  ✓
33.  $\frac{1}{1 + \sin^2 A} + \frac{1}{1 + \cos^2 A} + \frac{1}{1 + \sec^2 A} + \frac{1}{1 + \csc^2 A} = 2.$
34.  $\frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A} = \sin A + \cos A.$
35.  $\frac{\sin}{\csc A} + \frac{\cos A}{\sec A} = 1.$
36.  $\frac{\tan A - \cot A}{\tan A + \cot A} = \frac{2}{\csc^2 A} - 1.$
37.  $\frac{1 - \tan^2 \theta}{1 + \tan \theta} = \frac{\cos \theta - \sin}{\cos \theta}$
38.  $\frac{\sin^3 X}{\cos X - \cos^3 X} = \tan X.$
39.  $\frac{\tan X + \sec X - 1}{\tan X - \sec X + 1} = \tan X + \sec X.$
40.  $\frac{\tan \theta - \sin \theta}{\sin^3 \theta} = \frac{\sec \theta}{1 + \cos \theta}$

29. **Functions of  $(90^\circ + \theta)$ .** For future work we wish the functions of  $(90^\circ + \theta)$  in terms of  $\theta$ . We shall take  $\theta$  as acute; the results obtained, however, hold when  $\theta$  is of any magnitude.

In Figure 36, the angle  $XOB$  equals  $(90^\circ + \theta)$ ; the triangle with sides  $r'$ ,  $x'$ , and  $y'$  appears as in the figure.

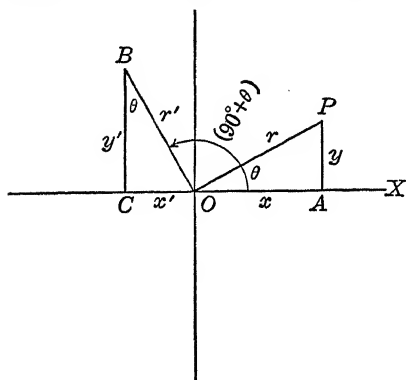


Figure 36.

From  $O$ , take  $OP$  perpendicular to  $OB$ , and of length  $r = r'$ . Drop a perpendicular from  $P$  to  $OX$  at  $A$ ; denote by  $r$ ,  $x$ , and  $y$  the sides of the right triangle thus formed. Then, the two triangles are congruent and we have:  $r = r'$ ;  $x = y'$ ; and  $y = -x'$ .

First we shall find:  $\sin(90^\circ + \theta)$ .

$$\sin(90^\circ + \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos \theta.$$

Similarly,

$$\cos(90^\circ + \theta) = \frac{x'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin \theta.$$

And:

$$\tan(90^\circ + \theta) = -\cot \theta,$$

$$\csc(90^\circ + \theta) = \sec \theta,$$

$$\sec(90^\circ + \theta) = -\csc \theta,$$

$$\cot(90^\circ + \theta) = -\tan \theta.$$

Observe that, except for the signs, the above results are exactly the same as those obtained in Section 11. Similar results obtain for the functions of  $(270^\circ \pm \theta)$ . Hence we have:

**Theorem.** Any function of  $(n90^\circ \pm \theta)$  when  $n$  is an odd positive integer is equal to the corresponding co-function of  $\theta$ , with the sign depending on the quadrant in which the angle lies.

We shall find particularly useful in Section 32 the following relation:

$$\cos(90^\circ + \theta) = -\sin \theta.$$

**30. Principal angle between two lines.** Consider the two *directed* lines  $AB$  and  $CD$ , intersecting at  $O$  (Figure 37). There are various angles from the *positive* direction of one line to the *positive* direction of the other, such as those indicated by 1, 2, and 3 on the figure. Of all such angles, there is one angle which is *positive* and less than  $180^\circ$ . We call this angle the *principal angle*. In Figure 37, it is angle 2.

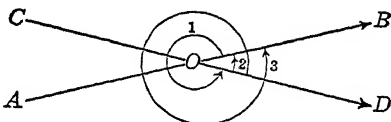


Figure 37.

**31. Projection.** Consider the directed line segment  $AB$  and the directed line  $CD$  in Figure 38. From  $A$  and  $B$ , respectively, drop  $AM$  and  $BN$  perpendicular to  $CD$ . The line segment  $MN$  is called the projection of  $AB$  on  $CD$ , and is written

$$\text{proj}_{CD} AB = MN.$$

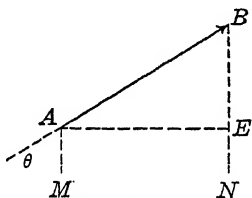


Figure 38.

Now, if  $AB$  is extended to meet  $CD$  and the principal angle is denoted by  $\theta$ , and if

$AE$  is drawn perpendicular to  $BN$ , it is evident that angle  $BAE = \theta$  and that  $AE = MN$ . Hence, since

$$\cos \theta = \frac{AE}{AB},$$

then:

$$MN = AE = AB \cos \theta.$$

From the above explanation we derive the first theorem on projection. The theorem is true regardless of the direction of the lines and the magnitude of  $\theta$ .

$$\text{proj}_{CD} AB = AB \cos \theta$$

**Theorem 1.** *The projection of a line segment on any line is equal to the product of the length of the line segment and the cosine of the principal angle between the lines.*

Consider the broken line  $OA, AB$  (Figure 39). Project  $OA, AB$ , and  $OB$  on  $CD$ . Then

$$\begin{aligned}\text{proj}_{CD} OA &= MN, \\ \text{proj}_{CD} AB &= NQ \text{ (NOT: } QN), \\ \text{proj}_{CD} OB &= MQ.\end{aligned}$$

But

$$\begin{aligned}MQ &= MN - QN \\ &= MN + NQ.\end{aligned}$$

Hence:

$$\text{proj}_{CD} OB = \text{proj}_{CD} OA + \text{proj}_{CD} AB.$$

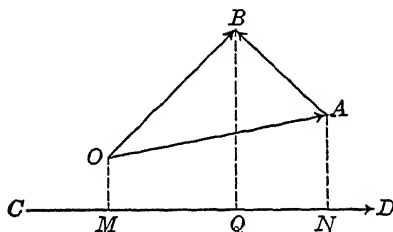


Figure 39.

From this computation, we have the second theorem on projection. This theorem may be extended for a broken line of any finite number of parts.

$$\text{proj } OB = \text{proj } OA + \text{proj } AB$$

**Theorem 2.** *The projection on any line of the broken line  $OA, AB$  is equal to the projection of  $OB$ .*

**32. Sine and cosine of the sum of two angles.** In the present section, we shall derive formulas for  $\sin(\alpha + \beta)$  and  $\cos(\alpha + \beta)$ ;  $\alpha$  and  $\beta$  may be any given angles. In Figure 40,  $\alpha$  and  $\beta$  are taken as acute, and are of such magnitude that their sum is less than  $90^\circ$ . It may be proved, however, that the formulas hold for angles of any magnitude.

Consider axes of coördinates with angles  $\alpha$  and  $\beta$  at the origin  $O$ , as in Figure 40. From any point  $P$  on the terminal

side of angle  $\beta$ , drop a perpendicular  $PA$  to the terminal side of angle  $\alpha$ . Extend  $PA$  to both axes to form angles as in the figure.

The right triangle  $OAP$  is the one upon which we shall focus our attention. The essence of our proof is to project the sides of this right triangle, first, on the  $x$ -axis and, then, on the  $y$ -axis. The first projection will give us  $\cos(\alpha + \beta)$ ; the second,  $\sin(\alpha + \beta)$ .

Projecting the directed sides of the right triangle  $OAP$  on the  $x$ -axis, we have, by the second theorem on projection:

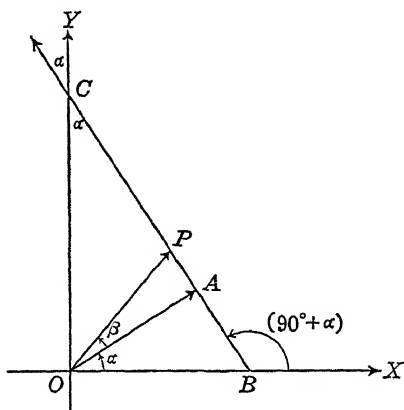


Figure 40.

$$\text{proj}_{Ox} OP = \text{proj}_{Ox} OA + \text{proj}_{Ox} AP.$$

By the first projection theorem, this becomes:

$$OP \cos(\alpha + \beta) = OA \cos \alpha + AP \cos(90^\circ + \alpha).$$

Or, since

$$\cos(90^\circ + \alpha) = -\sin \alpha,$$

we have:

$$OP \cos(\alpha + \beta) = OA \cos \alpha - AP \sin \alpha.$$

Dividing by  $OP$ , we have:

$$\cos(\alpha + \beta) = \cos \alpha \left( \frac{OA}{OP} \right) - \sin \alpha \left( \frac{AP}{OP} \right)$$

Or, since

$$\frac{OA}{OP} = \cos \beta$$

and

$$\frac{AP}{OP} = \sin \beta,$$

therefore:

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

In like fashion, we project the sides of the right triangle  $OAP$  on the  $y$ -axis and we have:

$$\text{proj}_{or} OP = \text{proj}_{or} OA + \text{proj}_{or} AP.$$

Substituting,

$$OP \cos [90^\circ - (\alpha + \beta)] = OA \cos (90^\circ - \alpha) + AP \cos \alpha,$$

or

$$OP \sin (\alpha + \beta) = OA \sin \alpha + AP \cos \alpha.$$

Dividing by  $OP$ , we have:

$$\sin (\alpha + \beta) = \sin \alpha \left( \frac{OA}{OP} \right) + \cos \alpha \left( \frac{AP}{OP} \right).$$

Or, finally,

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

**33. Tan  $(\alpha + \beta)$ .** A formula for  $\tan (\alpha + \beta)$  in terms of  $\tan \alpha$  and  $\tan \beta$  is derived as follows:

$$\begin{aligned} \tan (\alpha + \beta) &= \frac{\sin (\alpha + \beta)}{\cos (\alpha + \beta)} \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}. \end{aligned}$$

Dividing each member of this last fraction by  $\cos \alpha \cos \beta$ , we have:

$$\tan (\alpha + \beta) = \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}}.$$

Therefore:

$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

## Problems

## Example

Find, by using one of the addition formulas,  $\sin 75^\circ$ .

$$\begin{aligned}\sin 75^\circ &= \sin (45^\circ + 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

1. By using ( $75^\circ = 45^\circ + 30^\circ$ ), find  $\cos 75^\circ$ .
2. Find  $\tan 75^\circ$ .
3. Verify the relations for the functions of  $90^\circ$ .
4. Verify the relations for the functions of  $180^\circ$ .
5. Verify:  $\cos (90^\circ + \theta) = -\sin \theta$ .
6. Verify:  $\sin (180^\circ + \theta) = -\sin \theta$ .
7. Prove:

$$\tan (45^\circ + A) = \frac{\cos A + \sin A}{\cos A - \sin A}.$$

8. If  $\alpha + \beta = 45^\circ$ , prove:  $(1 + \tan \alpha)(1 + \tan \beta) = 2$ .
9. Prove:

$$\cot (\alpha + \beta) = \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta}.$$

10. If  $\tan \alpha = \frac{1}{2}$  and  $\tan \beta = \frac{1}{4}$ , prove:  $\tan (\alpha + \beta) = \frac{5}{7}$ .
11. If  $\tan \alpha = \frac{5}{8}$  and  $\tan \beta = \frac{1}{11}$ , prove:  $(\alpha + \beta) = 45^\circ$  or  $225^\circ$ .
12. If  $\tan \alpha = m$  and  $\tan \beta = n$  (assuming  $\alpha$  and  $\beta$  acute),  
prove:

$$\cos (\alpha + \beta) = \frac{1 - mn}{\sqrt{(1 + m^2)(1 + n^2)}}.$$

13. If  $\tan \alpha = \frac{3}{4}$  and  $\tan \beta = \frac{1}{5}$  (assuming  $\alpha$  and  $\beta$  acute),  
find  $\sin (\alpha + \beta)$ .

14. With the material of Problem 13, find  $\cos (\alpha + \beta)$ .

15. If  $\tan \alpha = \frac{m}{m+1}$  and  $\tan \beta = \frac{1}{2m+1}$ , prove:

$$\tan (\alpha + \beta) = 1.$$

**34. Functions of the difference of two angles.** We wish formulas for the sine, the cosine, and the tangent of  $(\alpha - \beta)$ . Using Roman instead of Greek letters, we rewrite

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

Let  $x = \alpha$ , and  $y = -\beta$ . Then, substituting, we have:

$$\sin[\alpha + (-\beta)] = \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta).$$

Or, since

$$\cos(-\beta) = \cos \beta$$

and

$$\sin(-\beta) = -\sin \beta$$

we have:

$$\begin{aligned}\sin(\alpha - \beta) &= \sin \alpha \cos \beta + \cos \alpha (-\sin \beta) \\ &= \sin \alpha \cos \beta - \cos \alpha \sin \beta.\end{aligned}$$

Similar results obtain for  $\cos(\alpha - \beta)$  and  $\tan(\alpha - \beta)$ . Hence we have:

$\begin{aligned}\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ \tan(\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}\end{aligned}$
---

### Problems

1. Find:

(a)  $\sin 15^\circ$ .

(b)  $\cos 15^\circ$ .

(c)  $\tan 15^\circ$ .  $2 - \sqrt{3}$

2. Verify:

(a)  $\sin(180^\circ - \theta) = \sin \theta$ .

(b)  $\cos(360^\circ - \theta) = \cos \theta$ .

(c)  $\tan(360^\circ - \theta) = -\tan \theta$ .

(d)  $\cos(270^\circ - \theta) = -\sin \theta$ .

3. If  $\tan \alpha = \frac{1}{3}$  and  $\tan \beta = \frac{1}{4}$ , find  $\tan(\alpha - \beta)$ .  $\frac{1}{13}$

4. If  $\tan \alpha = \frac{1}{3}$  and  $\tan \beta = \frac{4}{3}$  (assuming  $\alpha$  and  $\beta$  acute), find  $\cos (\alpha - \beta)$ .

5. If  $\tan \alpha = (x + 1)$  and  $\tan \beta = (x - 1)$ , prove:

$$\cot (\alpha - \beta) = \frac{x}{2}.$$

6. Prove:

$$(a) \tan (A - 45^\circ) + \cot (A + 45^\circ) = 0.$$

$$(b) \cot (A - 45^\circ) + \tan (A + 45^\circ) = 0.$$

$$(c) \cos (A + 45^\circ) + \sin (A - 45^\circ) = 0.$$

$$(d) \cos (A - 45^\circ) - \sin (A + 45^\circ) = 0.$$

$$(e) \sin (A - 45^\circ) = \frac{\sin A - \cos A}{\sqrt{2}}.$$

**35. Functions of a double-angle.** We wish formulas for the sine, the cosine, and the tangent of a double-angle, say  $2\alpha$ . Hence, we proceed as follows: Replacing  $\beta$  by  $\alpha$  in our addition formula, we have:

$$\begin{aligned}\sin 2\alpha &= \sin (\alpha + \alpha) \\ &= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha.\end{aligned}$$

Or:

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

Similarly,

$$\begin{aligned}\cos 2\alpha &= \cos (\alpha + \alpha) \\ &= \cos \alpha \cos \alpha - \sin \alpha \sin \alpha.\end{aligned}$$

Or:

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

Then, since

$$\sin^2 \alpha + \cos^2 \alpha = 1,$$

we have also:

$$\cos 2\alpha = 2 \cos^2 \alpha - 1.$$

Or:

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$

Similarly,

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

*Example*

Find  $\sin 3A$ .

$$\begin{aligned}\sin 3A &= \sin (2A + A) \\ &= \sin 2A \cos A + \cos 2A \sin A \\ &= (2 \sin A \cos A) \cos A + (1 - 2 \sin^2 A) \sin A \\ &= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A \\ &= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A \\ &= 3 \sin A - 4 \sin^3 A\end{aligned}$$

The above example illustrates the extensive use that may be made of the addition formulas: by this device the functions of any integral multiple of an angle may be found.

**Problems**

1. By using ( $60^\circ = 2 \cdot 30^\circ$ ), verify values for  $\sin 60^\circ$ ,  $\cos 60^\circ$ , and  $\tan 60^\circ$ .
2. Similarly, verify the values for the above functions of  $90^\circ$ .
3. Prove:  $\cos 3A = 4 \cos^3 A - 3 \cos A$ .
4. By using the material in Problem 3, verify the value for  $\cos 90^\circ$ .
5. Given  $\sin A = \frac{3}{5}$ ,  $\cos A$  positive; find  $\tan 2A$ .  $\frac{24}{7}$
6. Given  $\cos A = \frac{1}{3}$ ,  $\tan A$  negative; find  $\sin 2A$ .  $-\frac{12}{65}$
7. Given  $\tan A = -\frac{2}{3}$ ,  $\sin A$  negative; find  $\cos 2A$ .
8. Prove:

$$(a) \sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$(b) \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$(c) \frac{\tan (A + B) + \tan (A - B)}{1 - \tan (A + B) \tan (A - B)} = \tan 2A.$$

$$(d) \frac{1 + \sin 2\theta}{\cos 2\theta} = \frac{\cot \theta + 1}{\cot \theta - 1}$$

$$(e) \tan 2A + \sec 2A = \frac{\cos A + \sin A}{\cos A - \sin A}.$$

$$(f) \csc 2\theta = \frac{\sec \theta \csc \theta}{2}.$$

$$(g) \cot A = \frac{\sin 2A}{1 - \cos 2A}$$

$$(h) \frac{2 \sin^3 X}{1 - \cos X} = 2 \sin X + \sin 2X.$$

$$(i) \sec 2\theta = \frac{\sec^2 \theta}{2 - \sec^2 \theta}$$

$$(j) 1 + \sec 2A + \tan 2A = \frac{2}{1 - \tan A}.$$

$$(k) 1 + \sin 2X = \frac{(1 + \tan X)^2}{1 + \tan^2 X}.$$

9. Prove:

$$(a) 1 + \tan 2A \tan A = \sec 2A.$$

$$(b) \sin 2A(\tan A + \cot A) = 2.$$

$$(c) \tan \theta + \cot \theta = 2 \csc 2\theta.$$

$$(d) \cot \theta - \tan \theta = 2 \cot 2\theta.$$

$$(e) \cos^4 \theta - \sin^4 \theta = \cos 2\theta.$$

$$(f) \sin 2\alpha \sin \alpha = (1 - \cos 2\alpha) \cos \alpha.$$

$$(g) \cos 4X = 1 - 8 \sin^2 X + 8 \sin^4 X.$$

**36. Functions of a half-angle.** We wish first a formula for the sine of a half angle. Let us take

$$\sin \frac{\alpha}{2}.$$

We rewrite one of our formulas for the cosine of  $2X$  thus:

$$\cos 2X = 1 - 2 \sin^2 X.$$

Now let

$$X = \frac{\alpha}{2}.$$

Then

$$2X = \alpha.$$

Hence we have:

$$\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2},$$

or:

$$2 \sin^2 \frac{\alpha}{2} = 1 - \cos \alpha,$$

or:

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}.$$

Or, finally,

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

To find an expression for the cosine of a half-angle,

$$\cos \frac{\alpha}{2},$$

we rewrite one of the other formulas for the cosine of  $2X$  thus:

$$\cos 2X = 2 \cos^2 X - 1.$$

Again let

$$X = \frac{\alpha}{2}.$$

Then

$$2X = \alpha;$$

and we have:

$$\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1,$$

$$2 \cos^2 \frac{\alpha}{2} = 1 + \cos \alpha,$$

$$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2}.$$

Finally:

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

Next,

$$\tan \frac{\alpha}{2} = \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \frac{\pm \sqrt{\frac{1 - \cos \alpha}{2}}}{\pm \sqrt{\frac{1 + \cos \alpha}{2}}}$$

Or:

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

The formula just above may be simplified as follows:

$$\begin{aligned} \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} &= \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha} \cdot \frac{1 + \cos \alpha}{1 + \cos \alpha}} \\ &= \sqrt{\frac{1 - \cos^2 \alpha}{(1 + \cos \alpha)^2}} \\ &= \sqrt{\frac{\sin^2 \alpha}{(1 + \cos \alpha)^2}} \\ &= \frac{\sin \alpha}{1 + \cos \alpha} \end{aligned}$$

Similarly, if we multiply both numerator and denominator of

$$\frac{1 - \cos \alpha}{1 + \cos \alpha}$$

by

$$1 - \cos \alpha,$$

we obtain:

$$\frac{1 - \cos \alpha}{\sin \alpha}.$$

Hence we have:

$$\boxed{\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}}$$

*Example*

Prove:

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}.$$

We might solve this problem as follows:

$$\begin{aligned} 2 \sin \frac{A}{2} \cos \frac{A}{2} &= 2 \cdot \pm \sqrt{\frac{1 - \cos A}{2}} \cdot \pm \sqrt{\frac{1 + \cos A}{2}} \\ &= 2 \cdot \pm \sqrt{\frac{1 - \cos^2 A}{4}} \\ &= 2 \cdot \pm \sqrt{\frac{\sin^2 A}{4}} \\ &= 2 \cdot \frac{\sin A}{2} \\ &= \sin A. \end{aligned}$$

However, at this point the skillful student will recognize that the original relation is the formula (disguised a bit) for the sine of a double-angle. If we let

$$A = 2X,$$

then

$$\frac{A}{2} = X;$$

and the relation

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

becomes:

$$\sin 2X = 2 \sin X \cos X.$$

Indeed,

$$\sin \frac{A}{2} = 2 \sin \frac{A}{4} \cos \frac{A}{4}$$

is true for the same reason.

In other words, if an equation assumes the *form* of a well-known formula and the angles have the proper relation,

one to another, the equation is true, regardless of the form in which the angles are expressed. Thus

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2}$$

will be recognized as one of our formulas for  $\cos 2\alpha$ .

The student should not be misled, however, into thinking that all the problems below are solved in a similar manner. They are not. The above fact was pointed out simply as being of use in some instances only.

### Problems

1. Given  $\sin A = \frac{3}{5}$ ,  $\tan A$  positive; find  $\sin \frac{A}{2}$ .
2. Given  $\cos A = \frac{2}{3}$ ,  $\sin A$  negative; find  $\tan \frac{A}{2}$ .
3. Given  $\tan A = -\frac{5}{12}$ ,  $A$  not in the second quadrant; find  $\cos \frac{A}{2}$ .
4. Given  $\csc A = -\frac{5}{4}$ ,  $A$  not in the fourth quadrant; find  $\tan \frac{A}{2}$ .
5. Given  $\cot A = \frac{4}{3}$ ,  $A$  not in the second quadrant; find  $\sin \frac{A}{2}$ .
6. Prove the following identities:
  - (a)  $1 + \tan A \tan \frac{A}{2} = \sec A$ .
  - (b)  $\sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{1 + \sin A}$ .
  - (c)  $\frac{1}{\csc A - \cot A} = \cot \frac{A}{2}$ .
  - (d)  $\frac{1 - \tan^2 (\theta/2)}{1 + \tan^2 (\theta/2)} = \cos \theta$ .
  - (e)  $\tan^2 \frac{A}{2} = \frac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A}$ .
  - (f)  $\cot \frac{A}{2} = \frac{\sin A + \sin 2A}{\cos A - \cos 2A}$ .

$$(g) \frac{1 + \tan (A/2)}{1 - \tan (A/2)} = \tan A + \sec A.$$

$$(h) \left( \cot \frac{X}{2} - \tan \frac{X}{2} \right)^2 \left( \cot X - 2 \cot 2X \right) = 4 \cot X.$$

$$(i) \left( 1 + \cot^2 \frac{A}{2} \right) \sin A \tan \frac{A}{2} = 2.$$

$$(j) \sin^2 \frac{\theta}{2} \left( \cot \frac{\theta}{2} - 1 \right)^2 = 1 - \sin \theta.$$

$$(k) \tan \frac{A}{2} \tan A + 1 = \tan A \cot \frac{A}{2} - 1.$$

$$(l) \tan \left( 45^\circ + \frac{\theta}{2} \right) = \cot \left( 45^\circ - \frac{\theta}{2} \right).$$

$$(m) \cot \frac{A}{2} - \tan \frac{A}{2} = 2 \cot \frac{A}{2}.$$

$$(n) \tan \frac{A}{2} = \frac{1 - \cos A + \sin A}{1 + \cos A + \sin A}.$$

$$(o) 1 - 2 \cot X \tan \frac{X}{2} - \tan^2 \frac{X}{2} = 0.$$

**37. Product formulas.** We wish a formula for

$$\sin P + \sin Q$$

expressed as a product. We proceed as follows: Let  $P = \alpha + \beta$ , and  $Q = \alpha - \beta$ . Then:

$$\begin{aligned} \sin P + \sin Q &= \sin (\alpha + \beta) + \sin (\alpha - \beta) \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= 2 \sin \alpha \cos \beta. \end{aligned}$$

Here is our product; but it is in terms of functions of  $\alpha$  and  $\beta$ , and we wish it to involve  $P$  and  $Q$ . Solving the original relations for  $\alpha$  and  $\beta$  by addition and subtraction, we have:

$$\begin{aligned} \alpha &= \frac{P + Q}{2}, \\ \beta &= \frac{P - Q}{2}. \end{aligned}$$

Hence we have:

$$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}.$$

In like manner, we derive the other three product formulas and collect the four below:

$$(1) \sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$(2) \sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

$$(3) \cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$(4) \cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$$


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The above formulas are particularly useful in the branch of mathematics called calculus.

### Example 1

Prove:

$$\sin 3A - \sin A = 2 \cos 2A \sin A.$$

Let

$$3A = P,$$

$$A = Q.$$

Then

$$\frac{P+Q}{2} = \frac{4A}{2} = 2A,$$

and

$$\frac{P-Q}{2} = \frac{3A-A}{2} = A.$$

Then, substituting in the second of the product formulas, we have immediately:

$$\sin 3A - \sin A = 2 \cos 2A \sin A.$$

### Example 2

Prove:

$$\begin{aligned} \sin 3A + \sin 5A \\ \cos 3A - \cos 5A \end{aligned} = \cot A.$$

Let

$$P = 3A,$$

$$Q = 5A.$$

Then

$$\frac{P + Q}{2} = 4A,$$

and

$$\frac{P - Q}{2} = -A.$$

When we substitute the first product formula, the numerator becomes:

$$\begin{aligned}\sin 3A + \sin 5A &= 2 \sin 4A \cos (-A) \\ &= 2 \sin 4A \cos A.\end{aligned}$$

When we substitute the fourth product formula, the denominator becomes:

$$\begin{aligned}\cos 3A - \cos 5A &= -2 \sin 4A \sin (-A) \\ &= -2 \sin 4A (-\sin A) \\ &= 2 \sin 4A \sin A.\end{aligned}$$

$$\begin{aligned}\text{Hence: } \frac{\sin 3A + \sin 5A}{\cos 3A - \cos 5A} &= \frac{2 \sin 4A \cos A}{2 \sin 4A \sin A} \\ &= \frac{\cos A}{\sin A} \\ &= \cot A.\end{aligned}$$

### Problems

Prove the following identities:

1.  $\sin 5A + \sin 3A = 2 \sin 4A \cos A.$
2.  $\cos 4A - \cos 2A = -2 \sin 3A \sin A.$
3.  $\sin 2A + \sin 4A + \sin 6A = 4 \cos A \cos 2A \sin 3A.$
4.  $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos$   
4A.

$$5. \frac{\sin 2A + \sin A}{\cos 2A + \cos A} = \tan \frac{3A}{2}.$$

$$6. \frac{\cos 2A - \cos 3A}{\sin 2A + \sin 3A} = \tan \frac{A}{2}.$$

$$7. \frac{\sin A + \sin B}{\cos A - \cos B} = \frac{\cos A + \cos B}{\sin B - \sin A}.$$

8.  $\frac{\cos 6A - \cos 4A}{\sin 6A + \sin 4A} = -\tan A.$   
 $\frac{\sin 7A + \sin 3A}{\cos 7A - \cos 3A} = -\cot 2A.$
10.  $\frac{\sin 5A - \sin A}{\cos 5A + \cos A} = \tan 2A.$
11.  $\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{1}{\sqrt{3}}.$
12.  $\frac{\sin 5A - 2 \sin 3A + \sin A}{\cos 5A - 2 \cos 3A + \cos A} = \tan 3A.$
13.  $\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A.$
14.  $\frac{\sin 8A - \sin 6A + \sin 4A - \sin 2A}{\cos 8A - \cos 6A + \cos 4A - \cos 2A} = -\cot 5A.$
15.  $\frac{\sin 3A + \sin 2A + \sin A}{\cos 3A + \cos 2A + \cos A} = \tan 2A.$
16.  $\frac{\sin (A + 2B) - 2 \sin (A + B) + \sin A}{\cos (A + 2B) - 2 \cos (A + B) + \cos A} = \tan (A + B).$
17.  $\frac{\sin (2A - 3B) + \sin 3B}{\cos (2A - 3B) + \cos 3B} = \tan A.$
18.  $\frac{\sin 47^\circ + \sin 73^\circ}{\cos 47^\circ + \cos 73^\circ} = \sqrt{3}.$
19.  $\frac{\sin 4A - \sin 2A}{\sin 4A + \sin 2A} = \frac{1 - 3 \tan^2 A}{3 - \tan^2 A}.$
20.  $\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}}.$

### Miscellaneous Problems

Prove the following identities:

1.  $\sin 5A \sin A = \sin^2 3A - \sin^2 2A.$
2.  $\sin \theta + \sin (\theta - 120^\circ) + \sin (60^\circ - \theta) = 0.$
3.  $\cos^4 \theta - \sin^4 \theta = 2 \cos^2 \theta - 1.$
4.  $(\sec A - \tan A)(\sec A + \tan A) = 1.$

5.  $\sin^6 \theta + \cos^6 \theta = 1 - 3 \sin^2 \theta \cos^2 \theta.$
6.  $\sin (n+1)\theta = \sin n\theta \cos \theta + \cos n\theta \sin \theta.$
7.  $\sin 4X = 8 \cos^3 X \sin X - 4 \cos X \sin X.$
8.  $\frac{\sin A + \cos A}{\cos A - \sin A} = \tan 2A + \sec 2A.$
9.  $\cot^2 \theta \left( \frac{\sec \theta - 1}{1 + \sin \theta} \right) + \sec^2 \theta \left( \frac{\sin \theta - 1}{1 + \sec \theta} \right) = 0.$
10.  $\frac{\tan^3 A}{1 + \tan^2 A} + \frac{\cot^3 A}{1 + \cot^2 A} = \frac{1 - 2 \sin^2 A \cos^2 A}{\sin A \cos A}$
11.  $\frac{1 + \sin 2X + \cos 2X}{1 + \sin 2X - \cos 2X} \cot X.$
12.  $\frac{\sin (A - B)}{\sin A \sin B} + \frac{\sin (B - C)}{\sin B \sin C} + \frac{\sin (C - A)}{\sin C \sin A} = 0.$
13.  $\cos^6 \theta - \sin^6 \theta = \cos 2\theta (1 -$
14.  $\cos^4 X + \sin^4 X = 1 - \sin^2 2X$
15.  $\frac{2}{(1 + \tan \theta)(1 + \cot \theta)} = \frac{\sin}{1 + \sin 2\theta}.$
16.  $\frac{\sin (A - C) + 2 \sin A + \sin (A + C)}{\sin (B - C) + 2 \sin B + \sin (B + C)} = \frac{\sin A}{\sin B}.$
17.  $\frac{1 + \cos A}{1 - \cos A} = (\csc A + \cot A)^2.$
18.  $\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = 2.$
19.  $\frac{\tan A}{\tan A - \tan 3A} + \frac{\cot A}{\cot A - \cot 3A} = 1.$
20.  $\frac{2 \sin 2A - \sin 4A}{2 \sin 2A + \sin 4A} = \tan^2 A.$
21.  $\frac{1 - \cos X}{\sin X} = \frac{\sin 2X}{2 \cos X + \cos 2X + 1}$
22.  $\frac{2 \sec^2 X}{2 \tan X + 1} - \frac{\sec^2 X}{\tan X + 2} = \frac{6}{4 + 5 \sin 2X}.$
23.  $\cos \frac{3A}{2} = \cos \frac{A}{2} (2 \cos A - 1).$

$$24. \frac{4 \sin \theta \cos \frac{\theta}{2}}{2 \sin \theta + \sin 2\theta} = \sec \frac{\theta}{2}.$$

$$25. \frac{\sin A + \cos B}{\sin A - \cos B} = \frac{\tan \left( \frac{A+B}{2} + 45^\circ \right)}{\tan \left( \frac{A-B}{2} - 45^\circ \right)}.$$

## CHAPTER VII

### SUPPLEMENTARY TOPICS\*

**38. Law of tangents.** In Section 27 when we were considering the solution of an oblique triangle by means of the law of cosines, it was pointed out that there were additional formulas better adapted to logarithmic use but that we felt them to be unnecessary. However, since some authorities prefer them, we shall derive such formulas and include them in this chapter.

The first of these formulas is called the *law of tangents*. We shall proceed to its derivation.

Given an oblique triangle  $ABC$ , we have, from the law of sines,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

By the theory of proportion, this becomes:

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

From the first two product formulas, the right-hand side becomes:

$$\frac{2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}}{2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}} = \frac{\tan \frac{A - B}{2}}{\tan \frac{A + B}{2}}.$$

Or, we have the *law of tangents*:

\* This chapter may be omitted if the instructor does not wish to include the material in his course.

$$\frac{a-b}{a+b} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$$

By use of the law of tangents, we can solve a triangle if two sides and the included angle are given, as in the example below.

*Example*

Solve the triangle  $ABC$ ; given  $a = 2439$ ,  $b = 1036$ ,  $C = 38^\circ 7'$ .

$$a - b = 1403$$

$$a + b = 3475$$

$$A + B = 180^\circ - C = 180^\circ - 38^\circ 8' = 141^\circ 52'$$

$$\frac{A+B}{2} = 70^\circ 56'$$

$$\therefore \tan \frac{A-B}{2} = \frac{(a-b) \tan \frac{A+B}{2}}{a+b}$$

Using logarithms, we continue:

$$\log \tan \frac{A-B}{2} = \log (a-b) + \log \tan \frac{A+B}{2} - \log (a+b)$$

$$= \log 1403 + \log \tan 70^\circ 56' - \log 3475$$

$$\log 1403 = 3.1470$$

$$\log \tan 70^\circ 56' = .4614$$

$$\log \text{numerator} = 3.6084$$

$$\log 3475 = 3.5410$$

$$\log \tan \frac{A-B}{2} = .0674$$

$$\frac{A-B}{2} = 49^\circ 26'$$

$$\frac{A+B}{2} = 70^\circ 56'$$

But, since

by addition,

$$\therefore A = 120^\circ 22';$$

and, by subtraction,  $\therefore B = 21^\circ 30'$ .

Side  $c$  may now be found by the law of sines.

### Problems

Using the law of tangents, solve the following triangles:

1.  $a = 28.43$ ,  $b = 16.92$ ,  $C = 40^\circ 9'$ .  $18.47$ ;  $78^\circ 7'$ ;  $61^\circ 44'$
2.  $b = 623.1$ ,  $c = 420.3$ ,  $A = 62^\circ 42'$ .  $569.8$ ;  $76^\circ 21'$ ;  $40^\circ 57'$
3.  $c = 53.28$ ,  $a = 33.93$ ,  $B = 63^\circ 24'$ .  $48.71$ ;  $38^\circ 32'$ ;  $76^\circ 4'$
4.  $a = 419.2$ ,  $b = 300.3$ ,  $C = 53^\circ 18'$ .  $339.8$ ;  $81^\circ 35'$ ;  $45^\circ 7'$
5.  $a = 60.66$ ,  $b = 70.34$ ,  $C = 46^\circ 26'$ .  $45.13$ ;  $56^\circ 46'$ ;  $76^\circ 48'$

**39. Tangent of a half-angle in terms of the sides of a given triangle.** We shall now derive formulas to be used in solving a triangle when three sides are given.

Given the triangle  $ABC$ . Let

$$\frac{a + b + c}{2} = s$$

then:

$$s - a = \frac{b + c - a}{2},$$

$$s - b = \frac{c + a - b}{2},$$

$$s - c = \frac{a + b - c}{2}.$$

By the  $\tan \frac{A}{2}$  formula and the law of cosines:

$$\begin{aligned} \tan \frac{A}{2} &= \sqrt{\frac{1 - \cos A}{1 + \cos A}} \\ &= \sqrt{\frac{1 - \frac{b^2 + c^2 - a^2}{2bc}}{1 + \frac{b^2 + c^2 - a^2}{2bc}}} \end{aligned}$$

Or, by substitution,

$$\begin{aligned}
 \tan \frac{A}{2} &= \sqrt{\frac{2bc - b^2 - c^2 + a^2}{2bc + b^2 + c^2 - a^2}} \\
 &= \sqrt{\frac{a^2 - (b - c)^2}{(b + c)^2 - a^2}} \\
 &= \sqrt{\frac{(a - b + c)(a + b - c)}{(b + c + a)(b + c - a)}} \\
 &= \sqrt{\frac{(2)(s - b)(2)(s - c)}{(2s)(2)(s - a)}} \\
 \therefore \tan \frac{A}{2} &= \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}
 \end{aligned}$$

To make this expression more symmetrical, we write it :

$$\tan \frac{A}{2} = \sqrt{\frac{(s - a)(s - b)(s - c)}{s(s - a)^2}},$$

or

$$\tan \frac{A}{2} = \frac{1}{s - a} \sqrt{(s - a)(s - b)(s - c)}$$

Now let

$$\boxed{\sqrt{\frac{(s - a)(s - b)(s - c)}{s}} = r}$$

and we have:

$$\boxed{\tan \frac{A}{2} = \frac{r}{s - a}}$$

Similarly,

$$\tan \frac{B}{2} = \frac{r}{s - b}$$

$$\tan \frac{C}{2} = \frac{r}{s - c}$$

We shall next derive a formula for the area of the triangle  $ABC$  in terms of the sides. From Problem 2 in Section 27, we found:

$$\text{area} = \frac{1}{2} bc \sin A.$$

Or:

$$\begin{aligned} (\text{area})^2 &= \frac{1}{4} (bc)^2 \sin^2 A \\ &= \frac{1}{4} (bc)^2 (1 - \cos^2 A) \\ &= \frac{1}{4} (bc)^2 (1 + \cos A)(1 - \cos A). \end{aligned}$$

Or:

$$\begin{aligned} \text{area} &= \frac{1}{2} bc \sqrt{(1 + \cos A)(1 - \cos A)} \\ &= \frac{1}{2} bc \sqrt{\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right) \left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)} \\ &= \frac{1}{2} bc \sqrt{\frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4(bc)^2}} \\ &= \sqrt{s(s - a)(s - b)(s - c)}. \end{aligned}$$

Therefore:

$$\text{area} = \sqrt{s(s - a)(s - b)(s - c)}$$

### *Example*

Solve the triangle  $ABC$ ; given  $a = 100$ ,  $b = 120$ ,  $c = 140$ .

$$2s = 360$$

$$s = 180$$

$$s - a = 80$$

$$s - b = 60$$

$$s - c = 40$$

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$$

$$\log r = \frac{1}{2} \left[ \log (s - a) + \log (s - b) + \log (s - c) - \log s \right]$$

$$\log (s - a) = \log 80 = 1.9031$$

$$\log (s - b) = \log 60 = 1.7782$$

$$\log (s - c) = \log 40 = \underline{1.6021}$$

$$\text{sum} = 5.2834$$

$$\log s = \log 180 = \underline{2.2553}$$

$$2 \log r = 3.0281$$

$$\therefore \log r = 1.5141$$

$$\begin{aligned} \log \tan \frac{A}{2} &= \log r - \log (s - a) \\ &= (1.5141 - 10) - 1.9031 \\ &= 9.6110 - 10 \end{aligned}$$

$$\frac{A}{2} = 22^\circ 13'$$

$$\therefore A = 44^\circ 26'$$

$$\begin{aligned} \log \tan \frac{B}{2} &= \log r - \log (s - b) \\ &= (1.5141 - 10) - 1.7782 \\ &= 9.7359 - 10 \end{aligned}$$

$$\frac{B}{2} = 28^\circ 34'$$

$$\therefore B = 57^\circ 8'$$

$$\begin{aligned} \log \tan \frac{C}{2} &= \log r - \log (s - c) \\ &= (1.5141 - 10) - 1.6021 \\ &= 9.9120 - 10 \end{aligned}$$

$$\frac{C}{2} = 39^\circ 14'$$

$$\therefore C = 78^\circ 28'$$

*Proof*

$$\begin{aligned} A + B + C &= 44^\circ 26' + 57^\circ 8' + 78^\circ 28' \\ &= 179^\circ 62' \\ &= 180^\circ 2'. \end{aligned}$$

$$\begin{aligned}\log \text{ area} &= \frac{1}{2} [\log (s-a) + \log (s-b) + \log (s-c) + \log s] \\ &= \frac{1}{2} (7.5387)\end{aligned}$$

$$\log \text{ area} = 3.76985$$

$$\therefore \text{ area} = 5887 \text{ square units}$$

### Problems

Solve the following triangles, and find the area of each:

1.  $a = 20.34$ ,  $b = 16.48$ ,  $c = 30.24$ .  $39^{\circ}12'$ ;  $30^{\circ}48'$ ;  $110^{\circ}$

2.  $a = 144$ ,  $b = 266$ ,  $c = 300$ .  $28^{\circ}40'$ ;  $62^{\circ}26'$ ;  $85^{\circ}54'$

3.  $a = 2743$ ,  $b = 3201$ ,  $c = 4002$ .  $43^{\circ}$ ;  $52^{\circ}44'$ ;  $84^{\circ}16'$

4.  $a = 200$ ,  $b = 400$ ,  $c = 500$ .  $22^{\circ}20'$ ;  $49^{\circ}28'$ ;  $108^{\circ}12'$

5.  $a = 42.81$ ,  $b = 22.03$ ,  $c = 30.22$ .  $109^{\circ}2'$ ;  $29^{\circ}6'$ ;  $41^{\circ}52'$

**40. Radius of the inscribed circle.** It is interesting to note that the quantity

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

associated with the triangle  $ABC$  is actually the numerical length of the radius of the circle inscribed in the given triangle. We prove this result as follows:

### Proof

From plane geometry, we know:

$$\text{area of } \triangle ABC = \frac{1}{2} r' \cdot P,$$

where  $r'$  is the radius of the inscribed circle and  $P$  is the perimeter of the triangle.

Since

$$P = 2s,$$

we have:

$$\text{area} = r' \cdot s.$$

From the formula for area derived in Section 39, we have:

$$\begin{aligned}\text{area} &= \sqrt{(s-a)(s-b)(s-c)s} \\ &= \sqrt{\frac{(s-a)(s-b)(s-c)s^2}{s}}\end{aligned}$$

$$\begin{aligned}
 &= s \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \\
 &= s \cdot r. \\
 \therefore r' &= r.
 \end{aligned}$$

We give below another proof that is independent of area. Consider the triangle  $ABC$ —with inscribed circle, and with bisectors of the angles meeting at the center—in Figure 41.

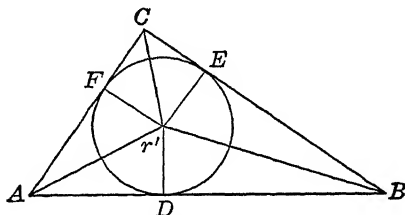


Figure 41.

Let  $r'$  be the radius of the circle. Then

$$\tan \frac{A}{2} = \frac{r'}{AD}.$$

We wish to prove:  $AD = s - a$ .

*Proof*

From plane geometry, we have:  $AD = AF$ ,  $DB = BE$ ,  $CF = CE$ .

Then  $AD + DB + CF = s$ .

Hence:  $AD = s - (DB + CF)$   
 $= s - (BE + CE);$   
 $= s - a.$

or by substitution,

Therefore:  $\tan \frac{A}{2} = \frac{r'}{s - a}.$

But since we know  $\tan \frac{A}{2} = \frac{r}{s - a},$   
 $\therefore r' = r.$

**41. Circular measure of an angle.** Frequently it is desirable, particularly in calculus, to express angles in units other than degrees. We do so by means of circular measure, as illustrated by Figure 42.

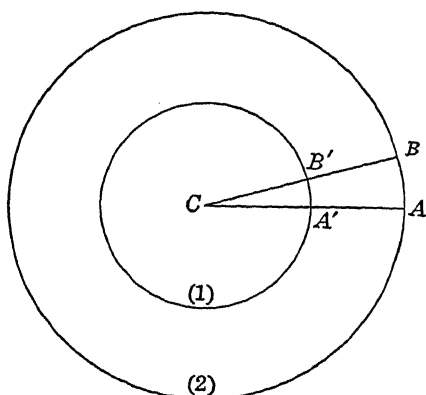


Figure 42.

Consider angle  $ACB$  and circles (1) and (2), with center at  $C$  and with radii  $r_1$  and  $r_2$ . Since

$$\frac{A'B'}{AB} = \frac{r_1}{r_2},$$

we have:

$$\frac{A'B'}{r_1} = \frac{AB}{r_2}.$$

In other words, the ratio

$$\frac{\text{length of arc}}{\text{radius of circle}}$$

is always the same for a given central angle. We call this ratio the *circular measure of an angle*, and we call the unit a *radian*. Hence we have:

$$\frac{\text{number of radians in central angle}}{\text{length of radius}} = \frac{\text{length of arc}}{\text{length of radius}}$$

To find the size of one radian, we substitute in the above formula and then have:

$$1 = \frac{\text{length of arc}}{\text{length of radius}}.$$

Hence:

$$\text{length of arc} = \text{length of radius}.$$

It is therefore evident that a radian is a central angle subtended by an arc equal in length to the radius of a circle.

There are as many radians in  $360^\circ$  as there are arcs of length  $r$  in a complete circumference. Since the circumference equals  $2\pi r$ , there are  $2\pi$  such arcs. Consequently there are  $2\pi$  radians in  $360^\circ$ ; or:

$$2\pi \text{ radians} = 360^\circ$$

Hence, to change degrees to radians, multiply the degrees by

$$\frac{\pi}{180}.$$

To change radians to degrees, multiply the radians by

$$\frac{180}{\pi}.$$

Thus:

$$60^\circ = 60 \cdot \frac{\pi}{180} \text{ radians} = \frac{\pi}{3} \text{ radians}.$$

Also:

$$\frac{2\pi}{3} \text{ radians} = \frac{2\pi}{3} \cdot \frac{180}{\pi} \text{ degrees} = 120^\circ$$

Since a radian is equivalent to

$$\frac{180^\circ}{3.14159},$$

a radian equals approximately

$$57^\circ 17' 45''.$$

Similarly, one degree equals

$$.01745 \dots \text{radian.}$$

### Problems

1. Change from degrees to radians:

- |                   |                   |                   |
|-------------------|-------------------|-------------------|
| (a) $60^\circ$ .  | (e) $270^\circ$ . | (i) $150^\circ$ . |
| (b) $30^\circ$ .  | (f) $240^\circ$ . | (j) $300^\circ$ . |
| (c) $120^\circ$ . | (g) $360^\circ$ . | (k) $180^\circ$ . |
| (d) $45^\circ$ .  | (h) $0^\circ$ .   | (l) $90^\circ$ .  |

2. Change from radians to degrees:

- |  |  |  |
|--|--|--|
| (a) $\frac{\pi}{2} \rightarrow 90^\circ$   | (e) $\frac{\pi}{6} \rightarrow 30^\circ$   | (i) $\frac{3\pi}{5} \rightarrow 108^\circ$ |
| (b) $\frac{2\pi}{3} \rightarrow 120^\circ$ | (f) $\frac{\pi}{3} \rightarrow 60^\circ$   | (j) $\frac{\pi}{4} \rightarrow 45^\circ$   |
| (c) $\frac{3\pi}{4} \rightarrow 135^\circ$ | (g) $\frac{5\pi}{3} \rightarrow 300^\circ$ | (k) $\frac{4\pi}{5} \rightarrow 144^\circ$ |
| (d) $2\pi \rightarrow 360^\circ$           | (h) $4\pi \rightarrow 720^\circ$           | (l) $3\pi \rightarrow 540^\circ$           |

**42. Summary of trigonometric formulas.** For convenience, we have collected in summary outline form the trigonometric formulas developed in the preceding text.

#### 1. Fundamental Identities.

$$\csc A = \frac{1}{\sin A}$$

$$\sec A = \frac{1}{\cos A}$$

$$\cot A = \frac{1}{\tan A}$$

$$\tan A = \frac{\sin A}{\cos A}$$

$$\cot A = \frac{\cos A}{\sin A}$$

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$1 + \cot^2 A = \csc^2 A$$

## 2. Addition and Subtraction Formulas.

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

## 3. Double-Angle Formulas.

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

## 4. Half-Angle Formulas.

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$$

$$\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha}$$

## 5. Product Formulas.

$$\sin P + \sin Q = 2 \sin \frac{P + Q}{2} \cos \frac{P - Q}{2}$$

$$\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

$$\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

### 6. Law of Sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

### 7. Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

### 8. Law of Tangents.

$$\frac{a-b}{a+b} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$$

### 9. Semi-Perimeter Formulas.

$$\tan \frac{A}{2} = \frac{r}{s-a}$$

$$\tan \frac{B}{2} = \frac{r}{s-b}$$

$$\tan \frac{C}{2} = \frac{r}{s-c}$$

$$s = \frac{a+b+c}{2}$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

### 10. Circular Measure.

$$\theta = \frac{l}{r}$$

The terms in this formula are interpreted:  $\theta$  = number of radians in a central angle,  $l$  = length of intercepted arc,  $r$  = length of radius.

$$\pi \text{ radians} = 180^\circ$$

### 11. Laws of Logarithms.

$$\log_b AB = \log_b A + \log_b B$$

$$\log_b \frac{A}{B} = \log_b A - \log_b B$$

$$\log_b A^n = n \log_b A$$

### 12. Projection Theorems.

$$\text{proj } AB + \text{proj } BC = \text{proj } AC$$

$$\text{proj}_{CD} AB = AB \cos \theta$$

The second theorem holds where  $\theta$  is the principal angle between  $AB$  and  $CD$ .



TABLE I  
LOGARITHMS TO FOUR PLACES

# FOUR-PLACE LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

# FOUR-PLACE LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

TABLE II  
TRIGONOMETRIC FUNCTIONS TO FOUR  
PLACES

# FOUR-PLACE TRIGONOMETRIC FUNCTIONS

Degrees	Sine Value Log	Tangent Value Log	Cotangent Value Log	Cosine Value Log	
0° 00'	.0000	.0000		1.0000 .0000	90° 00'
10	.0029 7.4637	.0029 7.4637	343.77 2.5363	1.0000 .0000	50
20	.0058 7.7648	.0058 7.7648	171.89 2.2352	1.0000 .0000	40
30	.0087 7.9408	.0087 7.9409	114.59 2.0591	1.0000 .0000	30
40	.0116 8.0658	.0116 8.0658	85.940 1.9342	.9999 .0000	20
50	.0145 8.1627	.0145 8.1627	68.750 1.8373	.9999 .0000	10
1° 00'	.0175 8.2419	.0175 8.2419	57.290 1.7581	.9998 9.9999	89° 00'
10	.0204 8.3088	.0204 8.3089	49.104 1.6911	.9998 9.9999	50
20	.0233 8.3668	.0233 8.3669	42.964 1.6331	.9997 9.9999	40
30	.0262 8.4179	.0262 8.4181	38.188 1.5819	.9997 9.9999	30
40	.0291 8.4637	.0291 8.4638	34.368 1.5362	.9996 9.9998	20
50	.0320 8.5050	.0320 8.5053	31.242 1.4947	.9995 9.9998	10
2° 00'	.0349 8.5428	.0349 8.5431	28.636 1.4569	.9994 9.9997	88° 00'
10	.0378 8.5776	.0378 8.5779	26.432 1.4221	.9993 9.9997	50
20	.0407 8.6097	.0407 8.6101	24.542 1.3899	.9992 9.9996	40
30	.0436 8.6397	.0437 8.6401	22.904 1.3599	.9990 9.9996	30
40	.0465 8.6677	.0466 8.6682	21.470 1.3318	.9989 9.9995	20
50	.0494 8.6940	.0495 8.6945	20.206 1.3055	.9988 9.9995	10
3° 00'	.0523 8.7188	.0524 8.7194	19.081 1.2806	.9986 9.9994	87° 00'
10	.0552 8.7423	.0553 8.7429	18.075 1.2571	.9985 9.9993	50
20	.0581 8.7645	.0582 8.7652	17.169 1.2348	.9983 9.9993	40
30	.0610 8.7857	.0612 8.7865	16.350 1.2135	.9981 9.9992	30
40	.0640 8.8059	.0641 8.8067	15.605 1.1933	.9980 9.9991	20
50	.0669 8.8251	.0670 8.8261	14.924 1.1739	.9978 9.9990	10
4° 00'	.0698 8.8436	.0699 8.8446	14.301 1.1554	.9976 9.9989	86° 00'
10	.0727 8.8613	.0729 8.8624	13.727 1.1376	.9974 9.9989	50
20	.0756 8.8783	.0758 8.8795	13.197 1.1205	.9971 9.9988	40
30	.0785 8.8946	.0787 8.8960	12.706 1.1040	.9969 9.9987	30
40	.0814 8.9104	.0816 8.9118	12.251 1.0882	.9967 9.9986	20
50	.0843 8.9256	.0846 8.9272	11.826 1.0728	.9964 9.9985	10
5° 00'	.0872 8.9403	.0875 8.9420	11.430 1.0580	.9962 9.9983	85° 00'
10	.0901 8.9545	.0904 8.9563	11.059 1.0437	.9959 9.9982	50
20	.0929 8.9682	.0934 8.9701	10.712 1.0299	.9957 9.9981	40
30	.0958 8.9816	.0963 8.9836	10.385 1.0164	.9954 9.9980	30
40	.0987 8.9945	.0992 8.9966	10.078 1.0034	.9951 9.9979	20
50	.1016 9.0070	.1022 9.0093	9.7882 .9907	.9948 9.9977	10
6° 00'	.1045 9.0192	.1051 9.0216	9.5144 .9784	.9945 9.9976	84° 00'
10	.1074 9.0311	.1080 9.0336	9.2553 .9664	.9942 9.9975	50
20	.1103 9.0426	.1110 9.0453	9.0098 .9547	.9939 9.9973	40
30	.1132 9.0539	.1139 9.0567	8.7769 .9433	.9936 9.9972	30
40	.1161 9.0648	.1169 9.0678	8.5555 .9322	.9932 9.9971	20
50	.1190 9.0755	.1198 9.0786	8.3450 .9214	.9929 9.9969	10
7° 00'	.1219 9.0859	.1228 9.0891	8.1443 .9109	.9925 9.9968	83° 00'
10	.1248 9.0961	.1257 9.0995	7.9530 .9005	.9922 9.9966	50
20	.1276 9.1060	.1287 9.1096	7.7704 .8904	.9918 9.9964	40
30	.1305 9.1157	.1317 9.1194	7.5958 .8806	.9914 9.9963	30
40	.1334 9.1252	.1346 9.1291	7.4287 .8709	.9911 9.9961	20
50	.1363 9.1345	.1376 9.1385	7.2687 .8615	.9907 9.9959	10
8° 00'	.1392 9.1436	.1405 9.1478	7.1154 .8522	.9903 9.9958	82° 00'
10	.1421 9.1525	.1435 9.1569	6.9682 .8431	.9899 9.9956	50
20	.1449 9.1612	.1465 9.1653	6.8269 .8342	.9894 9.9954	40
30	.1478 9.1697	.1495 9.1745	6.6912 .8255	.9890 9.9952	30
40	.1507 9.1781	.1524 9.1831	6.5606 .8169	.9886 9.9950	20
50	.1536 9.1863	.1554 9.1915	6.4348 .8085	.9881 9.9948	10
9° 00'	.1564 9.1943	.1584 9.1997	6.3138 .8003	.9877 9.9946	81° 00'
	Value Log Cosine	Value Log Cotangent	Value Log Tangent	Value Log Sine	Degrees

# FOUR-PLACE TRIGONOMETRIC FUNCTIONS

Degrees	Sine Value Log	Tangent Value Log	Cotangent Value Log	Cosine Value Log	
9° 00'	.1564 9.1943	.1584 9.1997	6.3138 .8003	.9877 9.9946	81° 00'
10	.1593 9.2022	.1614 9.2078	6.1970 .7922	.9872 9.9944	50
20	.1622 9.2100	.1644 9.2158	6.0844 .7842	.9868 9.9942	40
30	.1650 9.2176	.1673 9.2236	5.9758 .7764	.9863 9.9940	30
40	.1679 9.2251	.1703 9.2313	5.8708 .7687	.9858 9.9938	20
50	.1708 9.2324	.1733 9.2389	5.7694 .7611	.9853 9.9936	10
10° 00'	.1736 9.2397	.1763 9.2463	5.6713 .7537	.9848 9.9934	80° 00'
10	.1765 9.2468	.1793 9.2536	5.5764 .7464	.9843 9.9931	50
20	.1794 9.2538	.1823 9.2609	5.4845 .7391	.9838 9.9929	40
30	.1822 9.2606	.1853 9.2680	5.3955 .7320	.9833 9.9927	30
40	.1851 9.2674	.1883 9.2750	5.3093 .7250	.9827 9.9924	20
50	.1880 9.2740	.1914 9.2819	5.2257 .7181	.9822 9.9922	10
11° 00'	.1908 9.2806	.1944 9.2887	5.1446 .7113	.9816 9.9919	79° 00'
10	.1937 9.2870	.1974 9.2953	5.0658 .7047	.9811 9.9917	50
20	.1965 9.2934	.2004 9.3020	4.9894 .6980	.9805 9.9914	40
30	.1994 9.2997	.2035 9.3085	4.9152 .6915	.9799 9.9912	30
40	.2022 9.3058	.2065 9.3149	4.8430 .6851	.9793 9.9909	20
50	.2051 9.3119	.2095 9.3212	4.7729 .6788	.9787 9.9907	10
12° 00'	.2079 9.3179	.2126 9.3275	4.7046 .6725	.9781 9.9904	78° 00'
10	.2108 9.3238	.2156 9.3336	4.6382 .6664	.9775 9.9901	50
20	.2136 9.3296	.2186 9.3397	4.5736 .6603	.9769 9.9899	40
30	.2164 9.3353	.2217 9.3458	4.5107 .6542	.9763 9.9896	30
40	.2193 9.3410	.2247 9.3517	4.4494 .6483	.9757 9.9893	20
50	.2221 9.3466	.2278 9.3576	4.3897 .6424	.9750 9.9890	10
13° 00'	.2250 9.3521	.2309 9.3634	4.3315 .6366	.9744 9.9887	77° 00'
10	.2278 9.3575	.2339 9.3691	4.2747 .6309	.9737 9.9884	50
20	.2306 9.3629	.2370 9.3748	4.2193 .6252	.9730 9.9881	40
30	.2334 9.3682	.2401 9.3804	4.1653 .6196	.9724 9.9878	30
40	.2363 9.3734	.2432 9.3859	4.1126 .6141	.9717 9.9875	20
50	.2391 9.3786	.2462 9.3914	4.0611 .6086	.9710 9.9872	10
14° 00'	.2419 9.3837	.2493 9.3968	4.0108 .6032	.9703 9.9869	76° 00'
10	.2447 9.3887	.2524 9.4021	3.9617 .5979	.9696 9.9866	50
20	.2476 9.3937	.2555 9.4074	3.9136 .5926	.9689 9.9863	40
30	.2504 9.3986	.2586 9.4127	3.8667 .5873	.9681 9.9859	30
40	.2532 9.4035	.2617 9.4178	3.8208 .5822	.9674 9.9856	20
50	.2560 9.4083	.2648 9.4230	3.7760 .5770	.9667 9.9853	10
15° 00'	.2588 9.4130	.2679 9.4281	3.7321 .5719	.9659 9.9849	75° 00'
10	.2616 9.4177	.2711 9.4331	3.6891 .5669	.9652 9.9846	50
20	.2644 9.4223	.2742 9.4381	3.6470 .5619	.9644 9.9843	40
30	.2672 9.4269	.2773 9.4430	3.6059 .5570	.9636 9.9839	30
40	.2700 9.4314	.2805 9.4479	3.5656 .5521	.9628 9.9836	20
50	.2728 9.4359	.2836 9.4527	3.5261 .5473	.9621 9.9832	10
16° 00'	.2756 9.4403	.2867 9.4575	3.4874 .5425	.9613 9.9828	74° 00'
10	.2784 9.4447	.2899 9.4622	3.4495 .5378	.9605 9.9825	50
20	.2812 9.4491	.2931 9.4669	3.4124 .5331	.9596 9.9821	40
30	.2840 9.4533	.2962 9.4716	3.3759 .5284	.9588 9.9817	30
40	.2868 9.4576	.2994 9.4762	3.3402 .5238	.9580 9.9814	20
50	.2896 9.4618	.3026 9.4808	3.3052 .5192	.9572 9.9810	10
17° 00'	.2924 9.4659	.3057 9.4853	3.2709 .5147	.9563 9.9806	73° 00'
10	.2952 9.4700	.3089 9.4898	3.2371 .5102	.9555 9.9802	50
20	.2979 9.4741	.3121 9.4943	3.2041 .5057	.9546 9.9798	40
30	.3007 9.4781	.3153 9.4987	3.1716 .5013	.9537 9.9794	30
40	.3035 9.4821	.3185 9.5031	3.1397 .4969	.9528 9.9790	20
50	.3062 9.4861	.3217 9.5075	3.1084 .4925	.9520 9.9786	10
18° 00'	.3090 9.4900	.3249 9.5118	3.0777 .4882	.9511 9.9782	72° 00'
	Value Log Cosine	Value Log Cotangent	Value Log Tangent	Value Log Sine	Degrees

# FOUR-PLACE TRIGONOMETRIC FUNCTIONS

Degrees	Sine Value Log	Tangent Value Log	Cotangent Value Log	Cosine Value Log	
18° 00'	.3090 9.4900	.3249 9.5118	3.0777 .4882	.9511 9.9782	72° 00'
10	.3118 9.4939	.3281 9.5161	3.0475 .4839	.9502 9.9778	50
20	.3145 9.4977	.3314 9.5203	3.0178 .4797	.9492 9.9774	40
30	.3173 9.5015	.3346 9.5245	2.9887 .4755	.9483 9.9770	30
40	.3201 9.5052	.3378 9.5287	2.9600 .4713	.9474 9.9765	20
50	.3228 9.5090	.3411 9.5329	2.9319 .4671	.9465 9.9761	10
19° 00'	.3256 9.5126	.3443 9.5370	2.9042 .4630	.9455 9.9757	71° 00'
10	.3283 9.5163	.3476 9.5411	2.8770 .4589	.9446 9.9752	50
20	.3311 9.5199	.3508 9.5451	2.8502 .4549	.9436 9.9748	40
30	.3338 9.5235	.3541 9.5491	2.8239 .4509	.9426 9.9743	30
40	.3365 9.5270	.3574 9.5531	2.7980 .4469	.9417 9.9739	20
50	.3393 9.5306	.3607 9.5571	2.7725 .4429	.9407 9.9734	10
20° 00'	.3420 9.5341	.3640 9.5611	2.7475 .4389	.9397 9.9730	70° 00'
10	.3448 9.5375	.3673 9.5650	2.7228 .4350	.9387 9.9725	50
20	.3475 9.5409	.3706 9.5689	2.6985 .4311	.9377 9.9721	40
30	.3502 9.5443	.3739 9.5727	2.6746 .4273	.9367 9.9716	30
40	.3529 9.5477	.3772 9.5766	2.6511 .4234	.9356 9.9711	20
50	.3557 9.5510	.3805 9.5804	2.6279 .4196	.9346 9.9706	10
21° 00'	.3584 9.5543	.3839 9.5842	2.6051 .4158	.9336 9.9702	69° 00'
10	.3611 9.5576	.3872 9.5879	2.5826 .4121	.9325 9.9697	50
20	.3638 9.5609	.3906 9.5917	2.5605 .4083	.9315 9.9692	40
30	.3665 9.5641	.3939 9.5954	2.5386 .4046	.9304 9.9687	30
40	.3692 9.5673	.3973 9.5991	2.5172 .4009	.9293 9.9682	20
50	.3719 9.5704	.4006 9.6028	2.4960 .3972	.9283 9.9677	10
22° 00'	.3746 9.5736	.4040 9.6064	2.4751 .3936	.9272 9.9672	68° 00'
10	.3773 9.5767	.4074 9.6100	2.4545 .3900	.9261 9.9667	50
20	.3800 9.5798	.4108 9.6136	2.4342 .3864	.9250 9.9661	40
30	.3827 9.5828	.4142 9.6172	2.4142 .3828	.9239 9.9656	30
40	.3854 9.5859	.4176 9.6208	2.3945 .3792	.9228 9.9651	20
50	.3881 9.5889	.4210 9.6243	2.3750 .3757	.9216 9.9646	10
23° 00'	.3907 9.5919	.4245 9.6279	2.3559 .3721	.9205 9.9640	67° 00'
10	.3934 9.5948	.4279 9.6314	2.3369 .3686	.9194 9.9635	50
20	.3961 9.5978	.4314 9.6348	2.3183 .3652	.9182 9.9629	40
30	.3987 9.6007	.4348 9.6383	2.2998 .3617	.9171 9.9624	30
40	.4014 9.6036	.4383 9.6417	2.2817 .3583	.9159 9.9618	20
50	.4041 9.6065	.4417 9.6452	2.2637 .3548	.9147 9.9613	10
24° 00'	.4067 9.6093	.4452 9.6486	2.2460 .3514	.9135 9.9607	66° 00'
10	.4094 9.6121	.4487 9.6520	2.2286 .3480	.9124 9.9602	50
20	.4120 9.6149	.4522 9.6553	2.2113 .3447	.9112 9.9596	40
30	.4147 9.6177	.4557 9.6587	2.1943 .3413	.9100 9.9590	30
40	.4173 9.6205	.4592 9.6620	2.1775 .3380	.9088 9.9584	20
50	.4200 9.6232	.4628 9.6654	2.1609 .3346	.9075 9.9579	10
25° 00'	.4226 9.6259	.4663 9.6687	2.1445 .3313	.9063 9.9573	65° 00'
10	.4253 9.6286	.4699 9.6720	2.1283 .3280	.9051 9.9567	50
20	.4279 9.6313	.4734 9.6752	2.1123 .3248	.9038 9.9561	40
30	.4305 9.6340	.4770 9.6785	2.0965 .3215	.9026 9.9555	30
40	.4331 9.6366	.4806 9.6817	2.0809 .3183	.9013 9.9549	20
50	.4358 9.6392	.4841 9.6850	2.0655 .3150	.9001 9.9543	10
26° 00'	.4384 9.6418	.4877 9.6882	2.0503 .3118	.8988 9.9537	64° 00'
10	.4410 9.6444	.4913 9.6914	2.0353 .3086	.8975 9.9530	50
20	.4436 9.6470	.4950 9.6946	2.0204 .3054	.8962 9.9524	40
30	.4462 9.6495	.4986 9.6977	2.0057 .3023	.8949 9.9518	30
40	.4488 9.6521	.5022 9.7009	1.9912 .2991	.8936 9.9512	20
50	.4514 9.6546	.5059 9.7040	1.9768 .2960	.8923 9.9505	10
27° 00'	.4540 9.6570	.5095 9.7072	1.9626 .2928	.8910 9.9499	63° 00'
	Value Log Cosine	Value Log Cotangent	Value Log Tangent	Value Log Sine	Degrees

# FOUR-PLACE TRIGONOMETRIC FUNCTIONS

Degrees	Sine Value Log	Tangent Value Log	Cotangent Value Log	Cosine Value Log	
<b>27° 00'</b>	.4540 9.6570	.5095 9.7072	1.9626 .2928	.8910 9.9499	<b>63° 00'</b>
10	.4566 9.6595	.5132 9.7103	1.9486 .2897	.8897 9.9492	50
20	.4592 9.6620	.5169 9.7134	1.9347 .2866	.8884 9.9486	40
30	.4617 9.6644	.5206 9.7165	1.9210 .2835	.8870 9.9479	30
40	.4643 9.6668	.5243 9.7196	1.9074 .2804	.8857 9.9473	20
50	.4669 9.6692	.5280 9.7226	1.8940 .2774	.8843 9.9466	10
<b>28° 00'</b>	.4695 9.6716	.5317 9.7257	1.8807 .2743	.8829 9.9459	<b>62° 00'</b>
10	.4720 9.6740	.5354 9.7287	1.8676 .2713	.8816 9.9453	50
20	.4746 9.6763	.5392 9.7317	1.8546 .2683	.8802 9.9446	40
30	.4772 9.6787	.5430 9.7348	1.8418 .2652	.8788 9.9439	30
40	.4797 9.6810	.5467 9.7378	1.8291 .2622	.8774 9.9432	20
50	.4823 9.6833	.5505 9.7408	1.8165 .2592	.8760 9.9425	10
<b>29° 00'</b>	.4848 9.6856	.5543 9.7438	1.8040 .2562	.8746 9.9418	<b>61° 00'</b>
10	.4874 9.6878	.5581 9.7467	1.7917 .2533	.8732 9.9411	50
20	.4899 9.6901	.5619 9.7497	1.7796 .2503	.8718 9.9404	40
30	.4924 9.6923	.5658 9.7526	1.7675 .2474	.8704 9.9397	30
40	.4950 9.6946	.5696 9.7556	1.7556 .2444	.8689 9.9390	20
50	.4975 9.6968	.5735 9.7585	1.7437 .2415	.8675 9.9383	10
<b>30° 00'</b>	.5000 9.6990	.5774 9.7614	1.7321 .2386	.8660 9.9375	<b>60° 00'</b>
10	.5025 9.7012	.5812 9.7644	1.7205 .2356	.8646 9.9368	50
20	.5050 9.7033	.5851 9.7673	1.7090 .2327	.8631 9.9361	40
30	.5075 9.7055	.5890 9.7701	1.6977 .2299	.8616 9.9353	30
40	.5100 9.7076	.5930 9.7730	1.6864 .2270	.8601 9.9346	20
50	.5125 9.7097	.5969 9.7759	1.6753 .2241	.8587 9.9338	10
<b>31° 00'</b>	.5150 9.7118	.6009 9.7788	1.6643 .2212	.8572 9.9331	<b>59° 00'</b>
10	.5175 9.7139	.6048 9.7816	1.6534 .2184	.8557 9.9323	50
20	.5200 9.7160	.6088 9.7845	1.6426 .2155	.8542 9.9315	40
30	.5225 9.7181	.6128 9.7873	1.6319 .2127	.8526 9.9308	30
40	.5250 9.7201	.6168 9.7902	1.6212 .2098	.8511 9.9300	20
50	.5275 9.7222	.6208 9.7930	1.6107 .2070	.8496 9.9292	10
<b>32° 00'</b>	.5299 9.7242	.6249 9.7958	1.6003 .2042	.8480 9.9284	<b>58° 00'</b>
10	.5324 9.7262	.6289 9.7986	1.5900 .2014	.8465 9.9276	50
20	.5348 9.7282	.6330 9.8014	1.5798 .1986	.8450 9.9268	40
30	.5373 9.7302	.6371 9.8042	1.5697 .1958	.8434 9.9260	30
40	.5398 9.7322	.6412 9.8070	1.5597 .1930	.8418 9.9252	20
50	.5422 9.7342	.6453 9.8097	1.5497 .1903	.8403 9.9244	10
<b>33° 00'</b>	.5446 9.7361	.6494 9.8125	1.5399 .1875	.8387 9.9236	<b>57° 00'</b>
10	.5471 9.7380	.6536 9.8153	1.5301 .1847	.8371 9.9228	50
20	.5495 9.7400	.6577 9.8180	1.5204 .1820	.8355 9.9219	40
30	.5519 9.7419	.6619 9.8208	1.5108 .1792	.8339 9.9211	30
40	.5544 9.7438	.6661 9.8235	1.5013 .1765	.8323 9.9203	20
50	.5568 9.7457	.6703 9.8263	1.4919 .1737	.8307 9.9194	10
<b>34° 00'</b>	.5592 9.7476	.6745 9.8290	1.4826 .1710	.8290 9.9186	<b>56° 00'</b>
10	.5616 9.7494	.6787 9.8317	1.4733 .1683	.8274 9.9177	50
20	.5640 9.7513	.6830 9.8344	1.4641 .1656	.8258 9.9169	40
30	.5664 9.7531	.6873 9.8371	1.4550 .1629	.8241 9.9160	30
40	.5688 9.7550	.6916 9.8398	1.4460 .1602	.8225 9.9151	20
50	.5712 9.7568	.6959 9.8425	1.4370 .1575	.8208 9.9142	10
<b>35° 00'</b>	.5736 9.7586	.7002 9.8452	1.4281 .1548	.8192 9.9134	<b>55° 00'</b>
10	.5760 9.7604	.7046 9.8479	1.4193 .1521	.8175 9.9125	50
20	.5783 9.7622	.7089 9.8506	1.4106 .1494	.8158 9.9116	40
30	.5807 9.7640	.7133 9.8533	1.4019 .1467	.8141 9.9107	30
40	.5831 9.7657	.7177 9.8559	1.3934 .1441	.8124 9.9098	20
50	.5854 9.7675	.7221 9.8586	1.3848 .1414	.8107 9.9089	10
<b>36° 00'</b>	.5878 9.7692	.7265 9.8613	1.3764 .1387	.8090 9.9080	<b>54° 00'</b>
	Value Log Cosine	Value Log Cotangent	Value Log Tangent	Value Log Sine	Degrees

# FOUR-PLACE TRIGONOMETRIC FUNCTIONS

Degrees	Sine Value Log	Tangent Value Log	Cotangent Value Log	Cosine Value Log	
36° 00'	.5878 9.7692	.7265 9.8613	1.3764 .1387	.8090 9.9080	54° 00'
10	.5901 9.7710	.7310 9.8639	1.3680 .1361	.8073 9.9070	50
20	.5925 9.7727	.7355 9.8666	1.3597 .1334	.8056 9.9061	40
30	.5948 9.7744	.7400 9.8692	1.3514 .1308	.8039 9.9052	30
40	.5972 9.7761	.7445 9.8718	1.3432 .1282	.8021 9.9042	20
50	.5995 9.7778	.7490 9.8745	1.3351 .1255	.8004 9.9033	10
37° 00'	.6018 9.7795	.7536 9.8771	1.3270 .1229	.7986 9.9023	53° 00'
10	.6041 9.7811	.7581 9.8797	1.3190 .1203	.7969 9.9014	50
20	.6065 9.7828	.7627 9.8824	1.3111 .1176	.7951 9.9004	40
30	.6088 9.7844	.7673 9.8850	1.3032 .1150	.7934 9.8995	30
40	.6111 9.7861	.7720 9.8876	1.2954 .1124	.7916 9.8985	20
50	.6134 9.7877	.7766 9.8902	1.2876 .1098	.7898 9.8975	10
38° 00'	.6157 9.7893	.7813 9.8928	1.2799 .1072	.7880 9.8965	52° 00'
10	.6180 9.7910	.7860 9.8954	1.2723 .1046	.7862 9.8955	50
20	.6202 9.7926	.7907 9.8980	1.2647 .1020	.7844 9.8945	40
30	.6225 9.7941	.7954 9.9006	1.2572 .0994	.7826 9.8935	30
40	.6248 9.7957	.8002 9.9032	1.2497 .0968	.7808 9.8925	20
50	.6271 9.7973	.8050 9.9058	1.2423 .0942	.7790 9.8915	10
39° 00'	.6293 9.7989	.8098 9.9084	1.2349 .0916	.7771 9.8905	51° 00'
10	.6316 9.8004	.8146 9.9110	1.2276 .0890	.7753 9.8895	50
20	.6338 9.8020	.8195 9.9135	1.2203 .0865	.7735 9.8884	40
30	.6361 9.8035	.8243 9.9161	1.2131 .0839	.7716 9.8874	30
40	.6383 9.8050	.8292 9.9187	1.2059 .0813	.7698 9.8864	20
50	.6406 9.8066	.8342 9.9212	1.1988 .0788	.7679 9.8853	10
40° 00'	.6428 9.8081	.8391 9.9238	1.1918 .0762	.7660 9.8843	50° 00'
10	.6450 9.8096	.8441 9.9264	1.1847 .0736	.7642 9.8832	50
20	.6472 9.8111	.8491 9.9289	1.1778 .0711	.7623 9.8821	40
30	.6494 9.8125	.8541 9.9315	1.1708 .0685	.7604 9.8810	30
40	.6517 9.8140	.8591 9.9341	1.1640 .0659	.7585 9.8800	20
50	.6539 9.8155	.8642 9.9366	1.1571 .0634	.7566 9.8789	10
41° 00'	.6561 9.8169	.8693 9.9392	1.1504 .0608	.7547 9.8778	49° 00'
10	.6583 9.8184	.8744 9.9417	1.1436 .0583	.7528 9.8767	50
20	.6604 9.8198	.8796 9.9443	1.1369 .0557	.7509 9.8756	40
30	.6626 9.8213	.8847 9.9468	1.1303 .0532	.7490 9.8745	30
40	.6648 9.8227	.8899 9.9494	1.1237 .0506	.7470 9.8733	20
50	.6670 9.8241	.8952 9.9519	1.1171 .0481	.7451 9.8722	10
42° 00'	.6691 9.8255	.9004 9.9544	1.1106 .0456	.7431 9.8711	48° 00'
10	.6713 9.8269	.9057 9.9570	1.1041 .0430	.7412 9.8699	50
20	.6734 9.8283	.9110 9.9595	1.0977 .0405	.7392 9.8688	40
30	.6756 9.8297	.9163 9.9621	1.0913 .0379	.7373 9.8676	30
40	.6777 9.8311	.9217 9.9646	1.0850 .0354	.7353 9.8665	20
50	.6799 9.8324	.9271 9.9671	1.0786 .0329	.7333 9.8653	10
43° 00'	.6820 9.8338	.9325 9.9697	1.0724 .0303	.7314 9.8641	47° 00'
10	.6841 9.8351	.9380 9.9722	1.0661 .0278	.7294 9.8629	50
20	.6862 9.8365	.9435 9.9747	1.0599 .0253	.7274 9.8618	40
30	.6884 9.8378	.9490 9.9772	1.0538 .0228	.7254 9.8606	30
40	.6905 9.8391	.9545 9.9798	1.0477 .0202	.7234 9.8594	20
50	.6926 9.8405	.9601 9.9823	1.0416 .0177	.7214 9.8582	10
44° 00'	.6947 9.8418	.9657 9.9848	1.0355 .0152	.7193 9.8569	46° 00'
10	.6967 9.8431	.9713 9.9874	1.0295 .0126	.7173 9.8557	50
20	.6988 9.8444	.9770 9.9899	1.0235 .0101	.7153 9.8545	40
30	.7009 9.8457	.9827 9.9924	1.0176 .0076	.7133 9.8532	30
40	.7030 9.8469	.9884 9.9949	1.0117 .0051	.7112 9.8520	20
50	.7050 9.8482	.9942 9.9975	1.0058 .0025	.7092 9.8507	10
45° 00'	.7071 9.8495	1.0000 .0000	1.0000 .0000	.7071 9.8495	45° 00'
	Value Log Cosine	Value Log Cotangent	Value Log Tangent	Value Log Sine	Degrees

TABLE III  
SQUARES AND SQUARE ROOTS

# SQUARES AND SQUARE ROOTS

(Moving the decimal point *one* place in  $N$  requires a corresponding move of *two* places in  $N^2$ .)

N	N <sup>2</sup> 0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0001	.0004	.0009	.0016	.0025	.0036	.0049	.0064	.0081
0.1	.0100	.0121	.0144	.0169	.0196	.0225	.0256	.0289	.0324	.0361
0.2	.0400	.0441	.0484	.0529	.0576	.0625	.0676	.0729	.0784	.0841
0.3	.0900	.0961	.1024	.1089	.1156	.1225	.1296	.1369	.1444	.1521
0.4	.1600	.1681	.1764	.1849	.1936	.2025	.2116	.2209	.2304	.2401
0.5	.2500	.2601	.2704	.2809	.2916	.3025	.3136	.3249	.3364	.3481
0.6	.3600	.3721	.3844	.3969	.4096	.4225	.4356	.4489	.4624	.4761
0.7	.4900	.5041	.5184	.5329	.5476	.5625	.5776	.5929	.6084	.6241
0.8	.6400	.6561	.6724	.6889	.7056	.7225	.7396	.7569	.7744	.7921
0.9	.8100	.8281	.8464	.8649	.8836	.9025	.9216	.9409	.9604	.9801
1.0	1.000	1.020	1.040	1.061	1.082	1.103	1.124	1.145	1.166	1.188
1.1	1.210	1.232	1.254	1.277	1.300	1.323	1.346	1.369	1.392	1.416
1.2	1.440	1.464	1.488	1.513	1.538	1.563	1.588	1.613	1.638	1.664
1.3	1.690	1.716	1.742	1.769	1.796	1.823	1.850	1.877	1.904	1.932
1.4	1.960	1.988	2.016	2.045	2.074	2.103	2.132	2.161	2.190	2.220
1.5	2.250	2.280	2.310	2.341	2.372	2.403	2.434	2.465	2.496	2.528
1.6	2.560	2.592	2.624	2.657	2.690	2.723	2.756	2.789	2.822	2.856
1.7	2.890	2.924	2.958	2.993	3.028	3.063	3.098	3.133	3.168	3.204
1.8	3.240	3.276	3.312	3.349	3.386	3.423	3.460	3.497	3.534	3.572
1.9	3.610	3.648	3.686	3.725	3.764	3.803	3.842	3.881	3.920	3.960
2.0	4.000	4.040	4.080	4.121	4.162	4.203	4.244	4.285	4.326	4.368
2.1	4.410	4.452	4.494	4.537	4.580	4.623	4.666	4.709	4.752	4.796
2.2	4.840	4.884	4.928	4.973	5.018	5.063	5.108	5.153	5.198	5.244
2.3	5.290	5.336	5.382	5.429	5.476	5.523	5.570	5.617	5.664	5.712
2.4	5.760	5.808	5.856	5.905	5.954	6.003	6.052	6.101	6.150	6.200
2.5	6.250	6.300	6.350	6.401	6.452	6.503	6.554	6.605	6.656	6.708
2.6	6.760	6.812	6.864	6.917	6.970	7.023	7.076	7.129	7.182	7.236
2.7	7.290	7.344	7.398	7.453	7.508	7.563	7.618	7.673	7.728	7.784
2.8	7.840	7.896	7.952	8.009	8.066	8.123	8.180	8.237	8.294	8.352
2.9	8.410	8.468	8.526	8.585	8.644	8.703	8.762	8.821	8.880	8.940
3.0	9.000	9.060	9.120	9.181	9.242	9.303	9.364	9.425	9.486	9.548
3.1	9.610	9.672	9.734	9.797	9.860	9.923	9.986	10.05	10.11	10.18
3.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36
3.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40
4.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91

# SQUARES AND SQUARE ROOTS

(Moving the decimal point *one* place in  $N$  requires a corresponding move of *two* places in  $N^2$ .)

N	N <sup>2</sup> 0	1	2	3	4	5	6	7	8	9
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91
5.1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94
5.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98
5.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05
5.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25
5.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38
5.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52
5.8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69
5.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09
6.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32
6.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56
6.3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83
6.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43
6.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76
6.7	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	45.97	46.10
6.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47
6.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27
7.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70
7.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14
7.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61
7.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61
7.6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14
7.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68
7.8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25
7.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45
8.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08
8.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72
8.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39
8.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79
8.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52
8.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26
8.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03
8.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63
9.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46
9.2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30
9.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17
9.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.77	91.97
9.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90
9.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84
9.8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81
9.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80

## ANALYTIC GEOMETRY

## CHAPTER VIII

### COÖRDINATES

**43. Position of a point in a plane.** In Sections 16–18 we discussed directed distances, axes of coördinates, and quadrants. The student is advised to review these three sections immediately. From them it is quite evident that, for every point in a given plane, there is a unique set of two numbers called its coördinates; and that, conversely, for every set of two numbers, there is a unique point in the plane. In this chapter we shall be concerned with the coördinates of various points and with the algebraic or analytic quantities which will express, in terms of the coördinates, certain geometric properties associated with the points.

**44. Distance between two points.** The first formula that we shall derive is called the *distance* formula; it expresses the length of the line segment joining two points, in terms of their coördinates.

Given the two points  $P_1$  and  $P_2$ , with coördinates  $(x_1, y_1)$  and  $(x_2, y_2)$ , respectively (Figure 43); we desire a formula that will express the length  $P_1P_2$ .

After completing the right triangle  $P_1QP_2$  (Figure 43), we see that

$$P_1Q = M_1M_2 = OM_2 - OM_1 = x_2 - x_1.$$

Similarly,

$$QP_2 = y_2 - y_1.$$

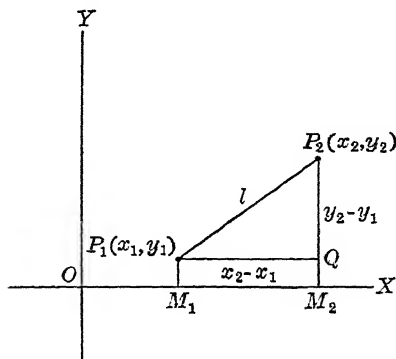


Figure 43.

Hence, by the law of Pythagoras,

$$P_1P_2 = l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Therefore we have the *distance* formula:

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

### Example

If  $P_1$  is  $(2, -3)$ , and  $P_2$  is  $(0, 4)$ , then:

$$\begin{aligned} P_1P_2 = l &= \sqrt{[0 - 2]^2 + [4 - (-3)]^2} \\ &= \sqrt{(-2)^2 + 7^2} \\ &= \sqrt{53}. \end{aligned}$$

The points  $P_1$  and  $P_2$  were taken in Figure 43 in most convenient positions. It is easy to show, however, that the formula is true regardless of the positions of  $P_1$  and  $P_2$ .

### Problems

1. Plot the following points:  $(2, -3)$ ,  $(0, -4)$ ,  $(4, 0)$ ,  $(-6, 2)$ .
2. What can be said regarding the coördinates of all points (a) on the  $x$ -axis? (b) on the  $y$ -axis? (c) on the line through the origin bisecting the first and the third quadrants? (d) on the line parallel to the  $x$ -axis and three units above it? and (e) on the line parallel to the  $y$ -axis and four units to the left of it?
3. Find the lengths of the sides of the triangle with the following points as vertices:  $(-2, 1)$ ,  $(3, 4)$ ,  $(2, -3)$ .
4. Do the same for the triangle with these vertices:  $(0, 4)$ ,  $(3, 0)$ ,  $(-1, -6)$ .
5. Show that the points  $(3, 4)$ ,  $(1, -2)$ , and  $(-3, 2)$  are the vertices of an isosceles triangle.
6. Show that the points  $(3, 2)$ ,  $(5, -1)$ , and  $(-3, -2)$  are the vertices of a right triangle.
7. Find whether or not the following points are the vertices of a right triangle:  $(-2, 0)$ ,  $(3, 5)$ ,  $(6, -2)$ .
8. Show that the points  $(-2, -3)$ ,  $(5, -4)$ ,  $(4, 1)$ , and  $(-3, 2)$  are the vertices of a parallelogram.

9. Show that the following points are the vertices of a parallelogram, and find whether or not the figure is a rectangle:  $(-3, 8)$ ,  $(-7, 6)$ ,  $(-3, -2)$ ,  $(1, 0)$ .

10. Show that the points  $(0, -3)$ ,  $(7, 2)$ ,  $(2, 9)$ , and  $(-5, 4)$  are the vertices of a square.

11. Show that the points  $(1, 4)$ ,  $(-2, 10)$ , and  $(3, 0)$  lie in a straight line.

12. Determine whether or not the points  $(0, -4)$ ,  $(3, 0)$ ,  $(5, 2)$  lie in a straight line.

45. Mid-point of a line segment. Given the line segment  $P_1P_2$ , with  $P$  the mid-point of this line segment (Figure 44); and the coördinates of  $P_1(x_1, y_1)$ , of  $P_2(x_2, y_2)$ , of  $P(x, y)$ . We wish to find the coördinates of  $P$  in terms of those of  $P_1$  and  $P_2$ .

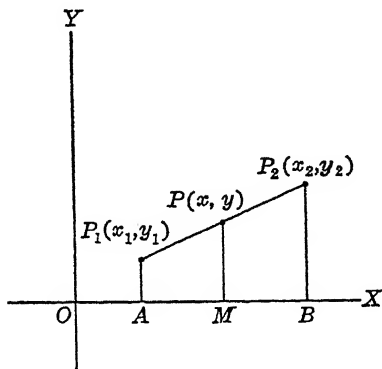


Figure 44.

As in Figure 44, drop perpendiculars  $P_1A$ ,  $PM$ , and  $P_2B$ . Then, since

$$P_1P = PP_2,$$

we have, from plane geometry,

$$AM = MB.$$

But

$$AM = x - x_1,$$

and

$$MB = x_2 - x.$$

Substituting,

$$x - x_1 = x_2 - x,$$

or

$$2x = x_1 + x_2.$$

Hence:

$$x = \frac{x_1 + x_2}{2}.$$

Similarly,

$$y = y_1 + y_2.$$

Therefore we have the *mid-point* formula:

$$\begin{array}{l} x = \frac{x_1 + x_2}{2} \\ y = \frac{y_1 + y_2}{2} \end{array}$$

*Example*

Find the coordinates of the point midway between  $(-1, 3)$  and  $(6, 5)$ .

$$x = \frac{x_1 + x_2}{2} = \frac{-1 + 6}{2} = \frac{5}{2}$$

$$y = \frac{y_1 + y_2}{2} = \frac{3 + 5}{2} = 4$$

46. Point that divides a line segment in a given ratio.

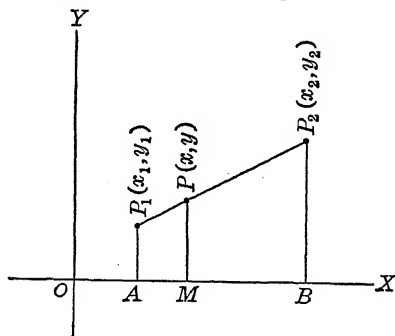


Figure 45.

Let us consider a line segment  $P_1P_2$  (Figure 45), with  $P$  a point on this line segment such that

$$\frac{\text{distance from } P_1 \text{ to } P}{\text{distance from } P \text{ to } P_2} = \frac{m}{n},$$

where

$$\frac{m}{n}$$

is any given ratio. Let us use the same coördinates as in Section 45. We wish to find the coördinates of  $P$  in terms of  $(x_1, y_1)$ , of  $(x_2, y_2)$ , and of  $m$  and  $n$ .

In Figure 45, we know, by plane geometry,

$$\frac{P_1P}{PP_2} = \frac{AM}{MB}.$$

Hence, by substitution,

$$\frac{m}{n} = \frac{x - x_1}{x_2 - x}.$$

Solving for  $x$ , we have:

$$x = \frac{mx_2 + nx_1}{m + n}$$

Likewise:

$$y = \frac{my_2 + ny_1}{m + n}$$

The above constitute the *ratio* formula.

In Figure 45, the segment  $P_1P_2$  is divided *internally*. If  $P$  lies on the segment extended, then we say the segment  $P_1P_2$  is divided *externally*. If the division is internal, the ratio is positive; if the division is external, the ratio is negative. The work will be simplified if the ratio is always considered as

$$\frac{\text{distance from } P_1 \text{ to } P}{\text{distance from } P \text{ to } P_2},$$

regardless of the position of  $P$ .

### Example 1

Given the segment joining  $(2, -1)$  and  $(8, 5)$ ; find the point of trisection nearer  $(2, -1)$ .

Call  $(2, -1)P_1$ , and  $(8, 5)P_2$ ; hence,  $x_1 = 2, y_1 = -1, x_2 = 8, y_2 = 5$ . The ratio is  $\frac{1}{2}$ ; hence,  $m = 1, n = 2$ .

Substituting in the ratio formula,

$$x = \frac{1 \cdot 8 + 2 \cdot 2}{1 + 2} = \frac{12}{3} = 4.$$

$$y = \frac{1 \cdot 5 + 2(-1)}{1 + 2} = \frac{3}{3} = 1.$$

Hence, the required point is the point  $(4, 1)$ .

### Example 2

The segment  $P_1P_2$  is extended half its length to  $P$  (Figure 46). Show that the ratio equals  $-3$ ; that is,

$$\frac{m}{n} = -3.$$

We have

$$\frac{m}{n} = \frac{P_1P}{PP_2}.$$

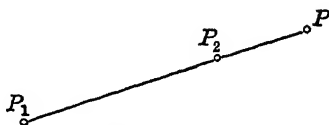


Figure 46.

Since  $P_2P$  is a unit,

$$P_1P = 3 \text{ units,}$$

$$PP_2 = -1 \text{ unit.}$$

Hence:

$$\frac{m}{n} = \frac{P_1P}{PP_2} = \frac{3}{-1} = -3.$$

### Problems

1. A triangle has the following points as vertices:  $A(0, 2)$ ,  $B(6, 0)$ , and  $C(4, 6)$ . Find:

- The coördinates of the mid-point of  $BC$ .
- The coördinates of the point two-thirds of the distance from  $A$  to the mid-point of  $BC$ .
- The coördinates of the mid-point of  $AB$ .
- The coördinates of the point two-thirds of the distance from  $C$  to the mid-point of  $AB$ .
- The length of the median through  $B$ .

2. Given the parallelogram with vertices at  $A(-2, -3)$ ,  $B(5, -4)$ ,  $C(4, 1)$ , and  $D(-3, 2)$ ; show that the coördinates of the mid-points of  $AC$  and  $BD$  are the same, and hence that the diagonals bisect each other.

3. Prove by the mid-point formula that the points  $(4, 12)$ ,  $(6, -2)$ ,  $(5, -10)$ , and  $(3, 4)$  form a parallelogram.

4. Three consecutive vertices of a parallelogram are  $(3, 0)$ ,  $(5, 2)$ , and  $(-2, 6)$ . Find the fourth vertex.

5. The segment from  $(-1, 2)$  to  $(3, -4)$  is doubled. Find the coördinates of the new end point.

6. The center of a circle is  $(3, 4)$ ; one point of the circle is  $(6, 8)$ . Find the coördinates of the other end of the diameter through this point.

7. Find the coördinates of the point that divides the segment from  $(0, -1)$  to  $(6, 3)$  in the ratio 2:5.

8. Find the ratio in which the point  $(2, -1)$  divides the segment from  $(6, 1)$  to  $(0, -2)$ .

9. Find the coördinates of the points that trisect the segment from  $(1, -4)$  to  $(3, 5)$ .

10. Find the coördinates of the points that divide the segment joining  $(2, -3)$  and  $(4, 1)$  into four equal parts.

11. Find the coördinates of the points that divide the segment from  $(1, 2)$  to  $(5, 8)$ , internally and externally, in the numerical ratio 3:2.

12.  $A$  is  $(-1, 3)$ , and  $B$  is  $(3, 6)$ . If  $AB$  is prolonged to  $C$ , a distance equal to three times its length, find the coördinates of  $C$ .

13. Find what point is reached by trebling the segment from  $(2, 0)$  to  $(3, 4)$ .

47. **Slope of a line.** The *slope* of a line is defined as the tangent of the angle

between the line and the  $x$ -axis, the angle being measured in counter-clockwise sense from the  $x$ -axis to the line. Thus, in Figure 47,  $\tan \theta_1$  is the slope of  $AB$ ;  $\tan \theta_2$ , the slope of  $CD$ .

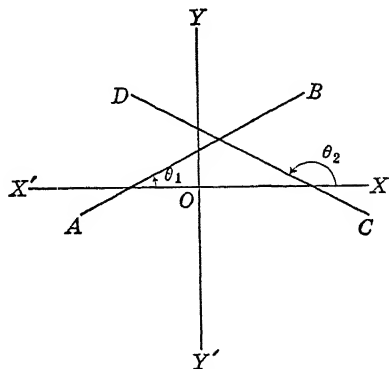


Figure 47.

Let us now consider a line  $AB$  passing through two points  $P_1$  and  $P_2$  (Figure 48), the coördinates of which are  $(x_1, y_1)$  and  $(x_2, y_2)$ , respectively. We wish to derive an expression for the slope of the line in terms of the coördinates of the two given points. Let us call the slope  $m$ . In Figure 48,

$$\angle CP_1P_2 = \theta.$$

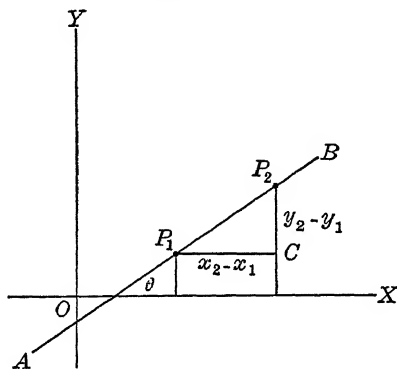


Figure 48.

Hence:

$$m = \tan \theta = \tan \angle CP_1P_2 = \frac{CP_2}{P_1C} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Thus we have the *slope* formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**48. Parallel and perpendicular lines.** If two lines are parallel, it is obvious that they have the same slope. Let us consider what relation, if any, exists between the slopes of two perpendicular lines.

Given two perpendicular lines  $AB$  and  $CD$ , making angles  $\theta_1$  and  $\theta_2$ , respectively, with the  $x$ -axis (Figure 49).

Let  $m_1 = \tan \theta_1$ , and  $m_2 = \tan \theta_2$ . Then:

$$\theta_1 = 90^\circ + \theta_2.$$

Therefore:

$$\tan \theta_1 = \tan (90^\circ + \theta_2) = -\cot \theta_2 = -\frac{1}{\tan \theta_2}$$

Hence, by substitution,

$$m_1 = -\frac{1}{m_2}$$

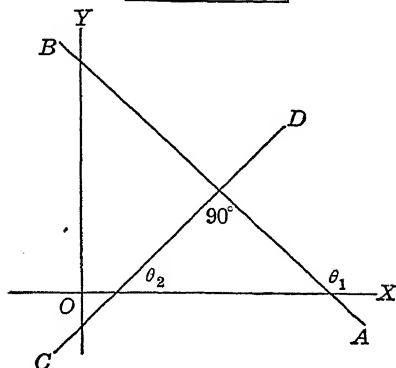


Figure 49.

That is, if two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other. The converse is true also.

**49. Angle between two lines.** Given two lines (1) and (2), in Figure 50; we may define the angle which line (1) makes with line (2) as the angle through which (2) must revolve in counter-clockwise sense to coincide with (1). In the figure, the angle is  $\alpha$ .

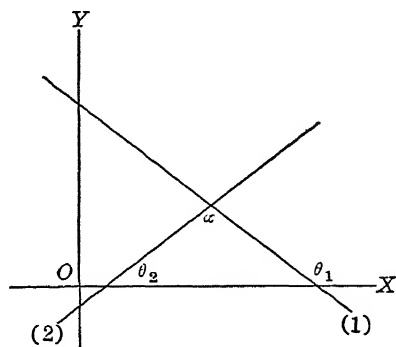


Figure 50.

We wish to express the tangent of  $\alpha$  in terms of the slopes  $m_1$  and  $m_2$  of the given lines.

Since

$$\alpha = \theta_1 - \theta_2,$$

hence:

$$\tan \alpha = \tan (\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}.$$

Therefore, by substitution,

$$\tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$$

### Problems

1. Find the slopes of the lines through:

- (a) (2, -1) and (3, 5).
- (b) (-3, -4) and (0, 6).
- (c) (1, 1) and (4, 4).

2. Find, by slopes, whether or not the following sets of points lie in the same straight line:

- (a) (4, 2), (0, -6), and (1, -4).
- (b) (0, 3), (9, 0), and (-3, 4).
- (c) (1, 1), (2, -1), and (1, -3).

3. Find the tangent of the angle between the lines the slopes of which are (a) -3 and  $\frac{1}{2}$ , respectively; (b) 3 and  $\frac{3}{4}$ ; (c)  $\frac{1}{3}$  and 2.

4. Show, by slopes, that the points (7, 2), (0, -3), (2, 9), and (-5, 4) are the vertices of a rectangle.

5. Three vertices of a parallelogram are (4, 1), (-3, 2), and (-2, -3). Find the fourth vertex. (NOTE: The student is expected to submit three possible solutions.)

6. Show, by slopes, that the following points form a parallelogram: (3, 0), (7, 3), (8, 5), and (4, 2).

7. A circle has its center at (3, 4). Find the slope of the tangent to the circle at (5, -2).

8. The base of a triangle passes through (2, -1) and (3, 2). Find the slope of the altitude.

9. Show that the diagonals of the square with vertices at (0, -3), (7, 2), (2, 9), and (-5, 4) are perpendicular.

10. Find whether or not the rectangle of Problem 4 is a square.

**50. Application of coördinates to plane geometry.** An interesting problem is that of proving theorems in plane geometry by means of coördinates. Two examples will illustrate the procedure.

Before we proceed, however, four considerations should be noted. First, to avoid a special case, we must use letters—not numbers—for coördinates. Second, we must use the most general type of figure for which the theorem is to be proved. Third, since the geometric properties of the figure are independent of its position, we can employ the most advantageous position for our particular figure. Hence, if our problem concerns a rectangle, we shall take the rectangle with two sides along the coördinate axes and with a vertex, therefore, at the origin. Finally, the coördinates and the relations between them add whatever further information is necessary to determine the type of figure.

#### Example 1

Prove that the diagonals of a rectangle are equal. In the present example, we shall use the rectangle in Figure 51.

Observe that, having taken  $O$  at  $(o, o)$ ,  $A$  at  $(a, o)$ , and  $B$  at  $(o, b)$ , and then having given  $C$  the coördinates  $(a, b)$ , we have made the figure a rectangle. We wish to prove that  $OC = AB$ . We use the distance formula:

$$OC = \sqrt{(a - o)^2 + (b - o)^2} = \sqrt{a^2 + b^2},$$

$$AB = \sqrt{(o - a)^2 + (b - o)^2} = \sqrt{a^2 + b^2}.$$

Hence:

$$OC = AB.$$

#### Example 2

Prove that the diagonals of a parallelogram bisect each other. We shall prove this theorem by showing that the mid-points of the two diagonals have the same coördinates and, therefore, coincide.

In Figure 52, we have taken one vertex at  $(o, o)$ ,  $A$  at  $(a, o)$ , and  $B$  at  $(b, c)$ . Then, since the figure is a parallelogram,  $C$  will have the coördinates  $(a + b, c)$ .

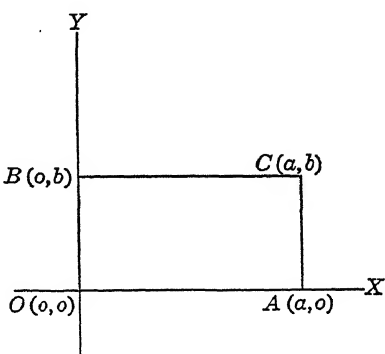


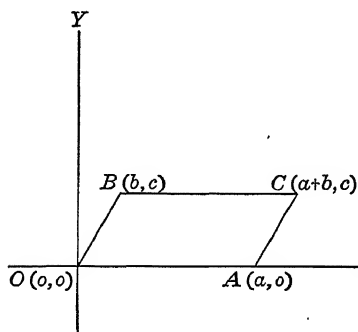
Figure 51.

We use the mid-point formula. The coördinates of the mid-point of  $OC$  are:

$$x = \frac{(a+b)+o}{2} = \frac{a+b}{2},$$

$$y = \frac{o+c}{2} = \frac{c}{2}.$$

The coördinates of the mid-point of  $AB$  are:



$$x = \frac{a+b}{2},$$

$$y = \frac{o+c}{2} = \frac{c}{2}.$$

The two mid-points have the same coördinates; hence, they coincide. Therefore,  $OC$  and  $AB$  bisect each other.

### Problems

Figure 52. Prove, by means of coördinates, the following geometric theorems:

1. The diagonals of a rectangle bisect each other.
2. The medians of a triangle meet in a point that is two-thirds of the way from a vertex to the mid-point of the opposite side.
3. The line joining the vertex of any right triangle with the mid-point of the hypotenuse is equal to half the hypotenuse.
4. The line joining the middle points of two sides of a triangle is equal to half the third side.
5. If the lines joining two vertices of a triangle to the middle points of the opposite sides are equal, the triangle is isosceles.
6. In any quadrilateral the lines joining the middle points of the opposite sides and the line joining the middle points of the diagonals meet in a point and bisect each other.
7. In any parallelogram  $ABCD$ , if  $M$  is the middle point of the side  $AB$ , the line  $MD$  and the diagonal  $AC$  trisect each other.
8. The sum of the squares of the medians of any triangle equals three-fourths of the sum of the squares of the sides.

9. The area of any triangle is four times the area of the triangle formed by joining the mid-points of the sides.

10. The lines joining the mid-points of adjacent sides of any rectangle form a rhombus.

11. The diagonals of a square are perpendicular.

12. The lines joining the middle points of the sides of any quadrilateral, taken in order, form a parallelogram.

13. The diagonals of a rhombus are perpendicular.

14. If the diagonals of a rectangle are perpendicular, the rectangle is a square.

15. A quadrilateral whose diagonals bisect each other at right angles is a rhombus.

16. The diagonals of an isosceles trapezoid are equal.

17. A trapezoid whose diagonals are equal is isosceles.

18. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

19. The distance between the mid-points of the non-parallel sides of a trapezoid is half the sum of the parallel sides.

20. The sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals.

## CHAPTER IX

### LOCUS

**51. Definition and equation of locus.** In the preceding chapter we considered geometric figures in which we were concerned with stationary points and their coördinates. Now we shall consider also the path described by a moving point. Let us call the path of a moving point a *curve*. (NOTE: A straight line is one form of a curve.) Usually the coördinates of any *general* point on a curve will be represented by  $(x, y)$ ; for any *particular* point, subscripts will be used:  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and so on.

If a point moves in such a way that it is constantly subject to a given condition that has been imposed, the path described by the point is defined as the *locus* of a point that moves subject to the given condition. The locus contains all those points, and only those points, that satisfy the given condition.

The circle is a well-known locus; it is the locus of a point moving so that its distance from a fixed point is always constant. Likewise, the perpendicular bisector of a line segment is the locus of a point moving so that it is always equidistant from the extremities of the line segment.

By the *equation of a locus*, we mean the equation which is satisfied by the coördinates of all those points, and only those points, that lie on the locus. The *equation of a curve* is defined similarly.

Thus it is evident that, *if a point lies on a curve, its coördinates must satisfy the equation of the curve; and conversely, if the coördinates of a point satisfy the equation of a curve, then the point must lie on the curve.* This idea is of fundamental importance and is applied constantly.

Furthermore, it is quite evident from the above that the intersection points of two or more curves are found by solving simultaneously the equations of those curves.

*Example 1*

Find the equation of the locus of a point moving so that its distance from the point  $(2, -3)$  is constantly equal to 6.

Call  $(x, y)$  the coördinates of the moving point. Then

$$\text{distance of } (x, y) \text{ from } (2, -3) = 6.$$

Substituting in the distance formula,

$$\sqrt{(x - 2)^2 + (y + 3)^2} = 6,$$

$$\text{or: } x^2 - 4x + 4 + y^2 + 6y + 9 = 36.$$

$$\text{Therefore: } x^2 + y^2 - 4x + 6y - 23 = 0.$$

The last is the equation of the locus. It is evident that this is the equation of a circle with center at  $(2, -3)$  and with radius 6.

*Example 2*

Find the equation of the locus of a point moving so that it is always equidistant from  $(1, 4)$  and  $(-3, 5)$ .

As before, call  $(x, y)$  the coördinates of the moving point. Then

$$\begin{aligned} \text{distance of } (x, y) \text{ from } (1, 4) \\ = \text{distance of } (x, y) \text{ from } (-3, 5). \end{aligned}$$

Substituting,

$$\sqrt{(x - 1)^2 + (y - 4)^2} = \sqrt{(x + 3)^2 + (y - 5)^2}.$$

$$\text{Therefore: } 8x - 2y + 17 = 0.$$

This is the equation of the perpendicular bisector of the line segment joining  $(1, 4)$  and  $(-3, 5)$ . The student may, by applying the mid-point formula, verify the fact that the mid-point of the given segment lies on the locus.

*Example 3*

Find the equation of the locus of a point moving so that its distance from the  $y$ -axis always equals its distance from the point  $(-1, 4)$ .

Since

$$\begin{aligned} &\text{distance of } (x, y) \text{ from } y\text{-axis} \\ &= \text{distance of } (x, y) \text{ from } (-1, 4), \end{aligned}$$

$$\text{substituting, } x = \sqrt{(x+1)^2 + (y-4)^2}.$$

Simplifying, therefore:

$$y^2 - 8y + 2x + 17 = 0.$$

#### Example 4

A variable triangle has as vertices the moving point  $(x, y)$ , the origin  $(0, 0)$ , and the point  $(6, 0)$ , as in Figure 53. The area always equals 10. Find the equation of the locus of  $(x, y)$ .

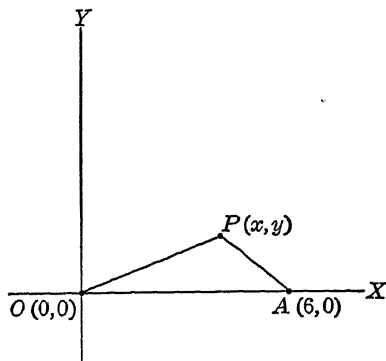


Figure 53.

The area of triangle  $OPA$  (Figure 53) is equal to

$$\frac{1}{2} \cdot OA \cdot h$$

Since  $OA = 6$ , and  $h = y$ , hence: area  $= \frac{1}{2} \cdot 6 \cdot y = 10$ ,

$$\text{or: } 3y = 10.$$

$$\text{Therefore: } y = \frac{10}{3}.$$

#### Problems

1. Find the equation of the locus of a point  $P$  moving as indicated in the following:

(a) Distance from  $(2, 4)$  equals 5.

(b) Distance from  $(-3, 5)$  equals distance from  $(1, -4)$ .

- (c) Distance from  $y$ -axis equals distance from  $(2, -1)$ .
- (d) Distance from  $x$ -axis is three times distance from  $y$ -axis.
- (e) Distance from  $y$ -axis is three times distance from  $x$ -axis.
- (f) Distance from origin is 6.
- (g) Distance from  $(-2, 3)$  equals twice distance from  $(1, 1)$ .
- (h) Distance from  $(5, 3)$  equals distance from  $x$ -axis.
- (i) Distance from  $y$ -axis equals 3.
- (j) Distance from  $x$ -axis equals  $-2$ .
- (k) Square of distance from origin is equal to sum of distance from  $x$ -axis plus distance from  $y$ -axis.
- (l) Sum of distances from  $(3, 0)$  and  $(-3, 0)$  equals 10.
- (m) Difference of distances from  $(5, 0)$  and  $(-5, 0)$  equals 8.
- (n) Product of distances from  $(2, 1)$  and  $(-1, 3)$  equals 5.
- (o) Sum of squares of distances from  $(4, -2)$  and  $(1, 1)$  equals 10.
- (p) Ratio of distances from  $(-3, 2)$  and  $(2, 7)$  equals  $\frac{3}{4}$ .

2. A variable triangle has as vertices the moving point  $(x, y)$ , the origin, and the point  $(4, 0)$ ; the area always equals 6. Find the equation of the locus of  $(x, y)$ .

3. The ends of a line of variable length are on two fixed perpendicular lines. Find the equation of the locus of the mid-point of this line if the area of the triangle thus formed is 20.

4. The base of a triangle is  $AB$ , where  $A$  is  $(-4, 0)$  and  $B$  is  $(2, 0)$ . Find the equation of the locus of the vertex  $P(x, y)$ , if the slope of  $AP$  is two units greater than the slope of  $BP$ .

5. The ends of the hypotenuse of a right triangle are  $(3, 4)$  and  $(-1, 6)$ . Find the equation of the locus of the vertex of the right angle.

6. The base of a triangle is  $AB$ , where  $A$  is  $(4, 0)$  and  $B$  is  $(8, 0)$ . Find the equation of the locus of the third vertex  $P(x, y)$ , if the median from  $A$  to  $BP$  is always three units in length.

## CHAPTER X

### THE STRAIGHT LINE

**52. Equations of lines parallel to the axes.** In the preceding chapter we considered equations of various curves derived from various loci conditions. In this chapter we shall concentrate on the straight line, the simplest type of locus. We shall first consider lines parallel to the coördinate axes.

In Figure 54, the line  $AB$  is parallel to the  $x$ -axis, and at a constant distance  $k$  from it. The equation of  $AB$  will be:

$$y = k$$

since all points whose ordinate is  $k$  will lie on the line  $AB$ , and the ordinate of all points on  $AB$  will be  $k$ . Likewise, the equation of a line parallel to the  $y$ -axis and at a constant distance  $k$  from it will be:

$$x = k$$

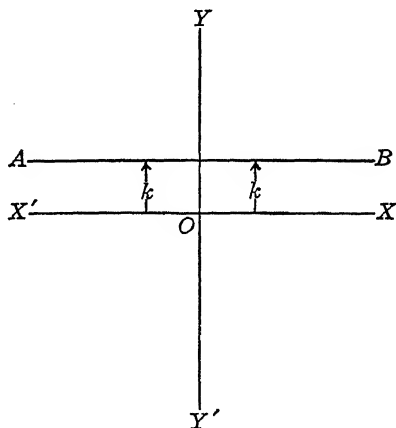


Figure 54.

**53. Point-slope form.** Let us now consider a line passing through a fixed point and having a fixed slope—that is, a fixed direction. We wish to find the equation of the line in terms of the fixed slope and the coördinates of the fixed point. By way of illustration, consider the line  $AB$  not parallel to either axis, as in Figure 55.

Call the fixed point  $(x_1, y_1)$ , and the slope  $m$ . Call  $(x, y)$  any other point on the line. Then, substituting in the slope formula (Section 47),

$$\frac{y - y_1}{x - x_1} = m.$$

Therefore we have the *point-slope* formula:

$$y - y_1 = m(x - x_1)$$

Conversely, we wish to show that all points whose coordinates satisfy the above equation lie on the line  $AB$  through  $(x_1, y_1)$  and with slope  $m$ .

Let  $(x_0, y_0)$  be the coordinates of any point satisfying the above equation (Figure 55). Then, substituting,

$$y_0 - y_1 = m(x_0 - x_1).$$

Or:

$$m = \frac{y_0 - y_1}{x_0 - x_1}.$$

But, by the slope formula, the expression

$$\frac{y_0 - y_1}{x_0 - x_1}$$

is the slope of the line  $CD$  through  $(x_0, y_0)$  and  $(x_1, y_1)$ . Since  $m$  equals this expression,  $m$  is the slope of the line  $CD$ . However,  $m$  is also the slope of the given line  $AB$ . Hence, since both lines,  $AB$  and  $CD$ , pass through  $(x_1, y_1)$ , and since they each have the same slope  $m$ , they must coincide; and thus,  $(x_0, y_0)$  must lie on the given line  $AB$ . Therefore the above point-slope formula is the equation of the line passing through  $(x_1, y_1)$  and with slope  $m$ .

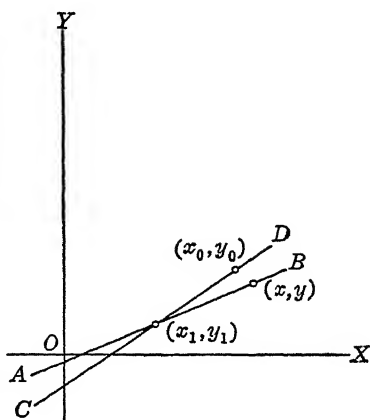


Figure 55.

This formula is called the *point-slope form* of the equation of a straight line, and is very useful in finding the equation of a line when its slope and a point on it are given.

### Problems

Write the equations of the following lines, and draw the figures:

1. Parallel to the  $y$ -axis, and at a distance of 3 units from it.
2. Parallel to the  $x$ -axis, and at a distance of  $-2$  units from it.
3. Through  $(2, -3)$ , and (a) parallel to  $OX$ ; (b) parallel to  $OY$ .
4. Through  $(4, -1)$ , and (a) parallel to  $x = 3$ ; (b) parallel to  $y = 6$ .
5. Through  $(3, 4)$ , and (a) parallel to  $x = -1$ ; (b) perpendicular to the same line.
6. (a) The  $x$ -axis; (b) the  $y$ -axis.
7. Through  $(2, -3)$ , and with slope 2.
8. Through  $(1, 4)$ , and with slope  $\frac{1}{2}$ .
9. Through  $(-1, 5)$ , and making an angle of  $30^\circ$  with the  $y$ -axis.
10. Through  $(2, 0)$ , and making an angle of  $135^\circ$  with the  $x$ -axis.
11. Through  $(3, -1)$ , and parallel to the line passing through  $(2, 4)$  and  $(-1, 3)$ .
12. Through  $(1, 6)$ , and perpendicular to the line passing through  $(3, 2)$  and  $(4, -3)$ .
13. Through  $(1, 1)$ , and parallel to the line passing through  $(4, 3)$  and  $(-1, 3)$ .
14. Through  $(3, 2)$ , and parallel to the line passing through  $(-2, 4)$  and  $(-2, 7)$ .
15. Through  $(o, b)$ , and with slope  $m$ .

**54. Slope-intercept form.** The  $x$ -intercept of a line is defined as the distance from the origin to the point where the line cuts the  $x$ -axis. The  $y$ -intercept is defined similarly.

Consider a line with slope  $m$ , and with  $y$ -intercept  $b$ . The line then passes through  $(o, b)$ . Substituting in the point-slope formula, we have:

$$y - b = m(x - o).$$

Thus we have the *slope-intercept* formula:

$$y = mx + b$$

This formula is called the *slope-intercept form*. The particular advantage of this formula is that it enables us to find the slope of a line immediately from its equation; we do this by solving for  $y$  and taking the resulting coefficient of  $x$  as the slope. Thus, given:  $3x + 2y - 6 = 0$ ,

$$y = -\frac{3}{2}x + 3.$$

Thus, comparing the coefficients of  $x$  in the example and in the formula, we see that

$$m = -\frac{3}{2}.$$

In this instance, the  $y$ -intercept is the resulting constant term. This fact is not of great importance, however, for we can always find the  $y$ -intercept by letting  $x$  equal zero and solving for  $y$ . Similarly, we might find the  $x$ -intercept by letting  $y$  equal zero and solving for  $x$ .

**55. Two-point form.** Given two points:  $(x_1, y_1)$  and  $(x_2, y_2)$ . We wish to find the equation of the line through these points.

Call the slope  $m$ . From the slope formula, we know that

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence, substituting in the point-slope formula, we have:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1).$$

Thus we have the *two-point* formula:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

**56. Intercept form.** Another formula interesting in this connection is the one called the *intercept form*.

We are given two intercepts: the  $x$ -intercept, called  $a$ ; and the  $y$ -intercept,  $b$ . Then, in the above formula,  $(x_1, y_1)$  is  $(a, 0)$ , and  $(x_2, y_2)$  is  $(0, b)$ . Hence:

$$\frac{y - 0}{x - a} = \frac{b - 0}{0 - a}.$$

Thus we have the *intercept* formula:

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

**57. General form of the equation of a straight line.** In the preceding paragraphs of this chapter, we found various forms of the equation of a straight line; in every instance the equation found was an equation of the first degree in  $x$  and  $y$ . It is evident that such will always be the case. Moreover, given conditions for a line—such as two points, two intercepts, or one point and slope—may be reduced to the case of a fixed point and slope, and then the point-slope formula may be applied. This procedure holds for all cases except when the line in question is parallel to the  $y$ -axis, and its equation consequently is of the form  $(x = k)$ . We now wish to show the converse; that is, every equation of the first degree in  $x$  and  $y$  is a straight line.

Consider the most general form of an equation of the first degree:

$$Ax + By + C = 0$$

where  $A$ ,  $B$ , and  $C$  are constants. If  $B$  is not zero, we may divide by  $B$ , and, after transposing terms, we have:

$$y = -\frac{A}{B} \cdot x - \frac{C}{B}.$$

This is of the form

$$y = mx + b,$$

which we know represents a line with slope  $m$  and with

$y$ -intercept  $b$ . Hence, we have a line with slope  $-\frac{A}{B}$ , and with  $y$ -intercept  $-\frac{C}{B}$ .

This solution, incidentally, tells us that the slope of the line

$$Ax + By + C = 0$$

is:

$$-\frac{\text{coefficient of } x}{\text{coefficient of } y};$$

and it gives us another method of finding the slope of a line if its equation is in the form

$$Ax + By + C = 0.$$

If  $B$  equals zero and  $A$  is not equal to zero, dividing by  $A$ , we have:

$$x = -\frac{C}{A}.$$

This equation is of the form

$$x = k,$$

which represents a line parallel to the  $y$ -axis.

Hence we say: *Every straight line is represented by an equation of the first degree, and every first degree equation represents a straight line.*

#### Example 1

Find the equation of the line passing through  $(-1, 2)$ , and parallel to  $3x - 4y + 2 = 0$ .

By solving for  $y$ , we first find the slope  $m$  of the given line.

$$-4y = -3x - 2$$

$$y = \frac{3}{4}x + \frac{1}{2}$$

$$\therefore m = \frac{3}{4}$$

The required line must have the same slope. Using the point-slope formula, we have:

$$y - 2 = \frac{3}{4}(x + 1),$$

which reduces to:

$$3x - 4y + 11 = 0.$$

*Example 2*

Find the equation of the line passing through  $(3, -4)$ , and perpendicular to  $2x + 3y - 6 = 0$ .

The slope of the given line is

$$m = -\frac{2}{3}.$$

The slope of the required line is

$$-\frac{1}{m},$$

and therefore

$$\frac{3}{2}.$$

Again using the point-slope formula, we have:

$$y + 4 = \frac{3}{2}(x - 3),$$

or:

$$3x - 2y - 17 = 0.$$

*Example 3*

The  $x$ -intercept of a line is three times the  $y$ -intercept. The line passes through  $(-6, 3)$ . Find its equation.

We use the intercept formula

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Since the  $x$ -intercept is three times the  $y$ -intercept,

$$a = 3b.$$

Hence:

$$\frac{x}{3b} + \frac{y}{b} = 1.$$

However, since the required line passes through  $(-6, 3)$ , (by the fundamental principle derived in Section 51) the coördinates  $(-6, 3)$  must satisfy the equation of the line. Hence, substituting, we have:

$$\frac{-6}{3b} + \frac{3}{b} = 1,$$

$$\frac{-2}{b} + \frac{3}{b} = 1,$$

$$b = 1.$$

Therefore:  $a = 3b = 3.$

Substituting further,  $\frac{x}{3} + \frac{y}{1} = 1.$

Therefore:  $x + 3y = 3.$

### Problems

1. Find the equations of the lines passing through:

- (a)  $(2, 0)$  and  $(-1, 3)$ .
- (b)  $(3, 3)$  and  $(4, -5)$ .
- (c)  $(-2, 6)$  and  $(2, -1)$ .
- (d)  $(0, 0)$  and  $(7, -3)$ .
- (e)  $(2, 4)$  and  $(2, 6)$ .
- (f)  $(3, -2)$  and  $(5, -2)$ .

2. Find the slopes of the following lines:

- (a)  $3x + 2y - 6 = 0.$
- (b)  $2x - y - 3 = 0.$
- (c)  $x + y - 2 = 0.$

3. Find the equations of the lines with:

- (a)  $x$ -intercept 3 and  $y$ -intercept  $-7$ .
- (b)  $x$ -intercept 2 and  $y$ -intercept 1.

4. Find the equation of the line passing through  $(-2, 4)$ , and parallel to  $y = 3x + 6$ .

5. Find the equation of the line passing through  $(2, -1)$ , and perpendicular to  $y = \frac{1}{3}x + 2$ .

6. Find the equation of the line passing through  $(1, 1)$ , and parallel to  $3x - 2y + 5 = 0$ .

7. Find the equation of the line passing through  $(-1, 3)$ , and perpendicular to  $x + y - 6 = 0$ .

8. Given  $2x - 3y - 6 = 0$ . Find: (a) the  $x$ -intercept; (b) the  $y$ -intercept; (c) whether or not  $(-1, 2)$  lies on the line.

9. A line passes through  $(3, -2)$ , and its  $x$ -intercept is 5. Find the equation of the line.

10. A line passes through  $(2, -4)$ , and its  $y$ -intercept is  $-2$ . Find the equation of the line.

11. A line passes through  $(2, 4)$ , and its  $x$ -intercept is twice its  $y$ -intercept. Find the equation of the line.

12. A line passes through  $(-1, 3)$ , and its  $y$ -intercept is four times its  $x$ -intercept. Find the equation of the line.

13. Find the equation of the line with slope  $\frac{1}{2}$  and with  $y$ -intercept  $-3$ .

14. Find the equation of the line with slope  $\frac{1}{3}$  and with  $x$ -intercept 2.

15. A circle is tangent to  $3x - 2y - 6 = 0$  at  $(2, 0)$ . Find the equation of the locus of its center.

**58. Normal form.** There is one other form of the equation of a straight line that will prove valuable, particularly in finding the distance from a line to a point. We proceed to its derivation as follows.

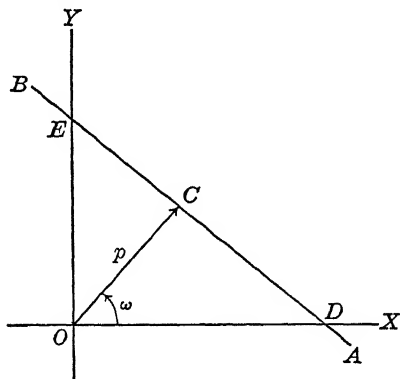


Figure 56.

The position of a straight line is completely determined if we know its perpendicular distance from the origin, and the angle this perpendicular makes with the  $x$ -axis. Thus, in Figure 56, the line  $AB$  is determined by  $p$  and  $\omega$ . Let the equation of  $AB$  be:  $y = mx + b$ . But since  $AB$  and  $OC$  are perpendicular,

$$m = -\frac{1}{\tan \omega} = -\cot \omega = -\frac{\cos \omega}{\sin \omega}.$$

Since  $\angle OEC = \angle \omega$ ,

$$b = OE = \frac{p}{\sin \angle OEC} = \frac{p}{\sin \omega}.$$

Hence, substituting,

$$y = -\frac{\cos \omega}{\sin \omega}x + \frac{p}{\sin \omega}.$$

Therefore, we have the *normal form*:

$$x \cos \omega + y \sin \omega - p = 0$$

We shall call  $p$  the *normal*,\* and shall consider  $p$  as always positive, its direction being from the origin to the line in question. If the line passes through the origin so that  $p = 0$ , we shall consider the arrow representing the direction of  $p$  as pointing in the first or the second quadrant (Figure 57). Unless the line passes through the origin,  $\omega$  may vary from  $0^\circ$  to  $360^\circ$ ; if the line passes through the origin,  $\omega$  varies only from  $0^\circ$  to  $180^\circ$ . For example, if the line passes through the origin,  $\omega = 200^\circ$  yields the same line as  $\omega = 20^\circ$ .

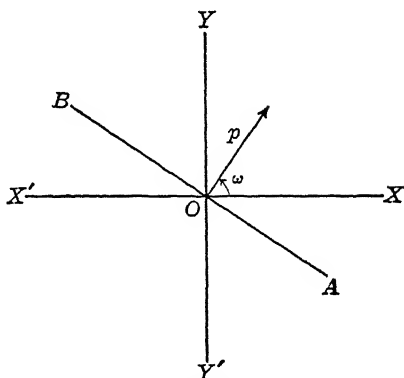


Figure 57.

Let us now consider the general form

$$(1) Ax + By + C = 0$$

and the normal form

$$(2) (\cos \omega)x + (\sin \omega)y - p = 0$$

---

\* Historically, *normal* means common or ordinary, but it is convenient in this text to use the term in its geometric connection.

to see what relation, if any, exists between the corresponding coefficients of  $x$  and  $y$  and the constant terms in the two equations. (We assume that the two equations represent the same line.) It is evident that the equation of a line is not affected by multiplying throughout by a constant. Thus,

$$6x + 4y + 14 = 0$$

represents the same line as

$$3x + 2y + 7 = 0.$$

Hence our problem may be stated: Find what constant, if any, which, multiplying equation (1), reduces it identically to equation (2).

Let us call the constant  $k$ . Equation (1) becomes:

$$(kA)x + (kB)y + (k)C = 0.$$

Comparing this with

$$(\cos \omega)x + (\sin \omega)y - p = 0,$$

we have the following three equations:

$$kA = \cos \omega,$$

$$kB = \sin \omega,$$

$$kC = -p.$$

Squaring and adding the first two equations, we have:

$$k^2(A^2 + B^2) = \cos^2 \omega + \sin^2 \omega = 1.$$

Therefore:

$$k = \frac{1}{\pm \sqrt{A^2 + B^2}}.$$

We now use the third equation to determine the sign of  $k$ . Since the quantity  $p$  must be positive,  $kC$  must be negative. Hence  $k$  and  $C$  must have opposite signs.

If  $C = 0$ , we use the relation  $kB = \sin \omega$ ; for, if  $C = 0$ , the line passes through the origin, in which case  $\omega$  varies only from  $0^\circ$  to  $180^\circ$  and  $\sin \omega$  remains always positive.

$kB$  must then be positive. Hence  $k$  and  $B$  must have like signs.

We may sum up this discussion as follows: To reduce the equation

$$Ax + By + C = 0$$

to the normal form, multiply throughout by

$$\frac{1}{\pm \sqrt{A^2 + B^2}},$$

and take the sign of the radical opposite to that of  $C$ . If  $C = 0$ , take the same sign as that of  $B$ .

#### *Example*

Reduce to the normal form:  $3x - 4y - 5 = 0$ .

$$A = 3$$

$$B = -4$$

$$\sqrt{A^2 + B^2} = 5$$

$$C = -5$$

$$\frac{3x - 4y - 5}{5} = 0$$

#### **Problems**

##### *Example*

Find the distance between the parallel lines (1)  $5x + 12y - 13 = 0$  and (2)  $5x + 12y - 39 = 0$ .

Reducing the lines to the normal form, we have:

$$(1) \frac{5x + 12y - 13}{13} = 0,$$

$$(2) \frac{5x + 12y - 39}{13} = 0.$$

The distance from the origin to the first line—that is,  $p_1$ —equals 1 unit; and the distance from the origin to the second line—that is,  $p_2$ —equals 3 units. Hence the distance between the lines is  $p_2$  minus  $p_1$ , or 2 units. (NOTE: These lines are on the same side of the origin. If they had been on opposite sides, we should have *added*  $p_1$  and  $p_2$ .)

1. Reduce to the normal form, and give the distance from the origin to the line in question:

- |                          |                        |
|--------------------------|------------------------|
| (a) $3x - 4y - 10 = 0.$  | (g) $2x + 3y - 4 = 0.$ |
| (b) $3x - 4y + 10 = 0.$  | (h) $x + y - 2 = 0.$   |
| (c) $5x + 12y - 26 = 0.$ | (i) $x - y = 0.$       |
| (d) $3x + 4y = 0.$       | (j) $x - 5 = 0.$       |
| (e) $3x - 4y = 0.$       | (k) $y + 6 = 0.$       |
| (f) $2x + y = 0.$        | (l) $y = 0$            |

2. Find the distance between the following parallel lines:

- |                        |     |                     |
|------------------------|-----|---------------------|
| (a) $3x - 4y - 10 = 0$ | and | $3x - 4y - 25 = 0.$ |
| (b) $2x - 4y - 3 = 0$  | and | $4x - 8y - 1 = 0.$  |
| (c) $3x - 4y - 5 = 0$  | and | $3x - 4y + 10 = 0.$ |

3. Find the equations of the following lines:

- (a) Distance from origin, 6; angle made by normal and  $x$ -axis,  $30^\circ$ .  
 (b) Distance from origin, 5; angle made by normal and  $x$ -axis,  $60^\circ$ .  
 (c) Distance from origin, 5; angle made by normal and  $x$ -axis,  $90^\circ$ .  
 (d) Distance from origin, 4; angle made by normal and  $x$ -axis,  $150^\circ$ .  
 (e) Distance from origin, 4; angle made by normal and  $x$ -axis,  $225^\circ$ .  
 (f) Distance from origin, 5; angle made by line and  $x$ -axis,  $45^\circ$ .  
 (g) Distance from origin, 5; angle made by line and  $x$ -axis,  $60^\circ$ .  
 (h) Distance from origin, 6; slope,  $-\frac{3}{4}$ .  
 (i) Distance from origin, 5; parallel to  $2x - 3y + 6 = 0$ .  
 (j) Distance from origin, 4; perpendicular to  $x + 2y - 4 = 0$ .

**59. Distance from a line to a point.** We shall now employ the normal form to find an expression for the perpendicular distance from a line to a point, when we are given the equation of the line and the coördinates of the point.

Let the equation of  $AB$  (Figure 58) be:

$$x \cos \omega + y \sin \omega - p = 0,$$

and the coördinates of  $P$ ,  $(x_1, y_1)$ . Let the perpendicular distance from  $AB$  to  $P$  be represented by  $d$ . Draw  $CD$  through  $(x_1, y_1)$  and parallel to  $AB$ . Call the distance  $OE$ ,  $p'$ . Then, the equation of  $CD$  is:

$$x \cos \omega + y \sin \omega - p' = 0,$$

or, since  $p' = p + d$ ,

$$x \cos \omega + y \sin \omega - (p + d) = 0.$$

Then, since  $P$  lies on  $CD$ , its coördinates  $(x_1, y_1)$  must satisfy the equation of  $CD$ .

Hence:

$$x_1 \cos \omega + y_1 \sin \omega - (p + d) = 0.$$

Therefore we have the formula for the distance from a line to a point:

$$d = x_1 \cos \omega + y_1 \sin \omega - p$$

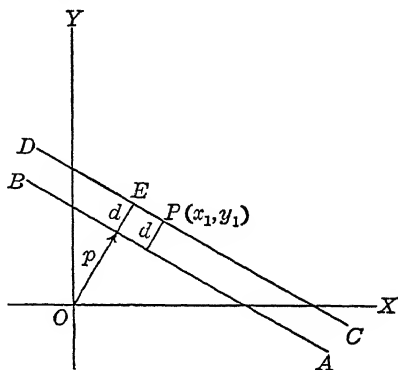


Figure 58.

Thus, to find the perpendicular distance from a line to a point, reduce the equation of the line to the normal form, and substitute for  $x$  and  $y$  the coördinates of the point.

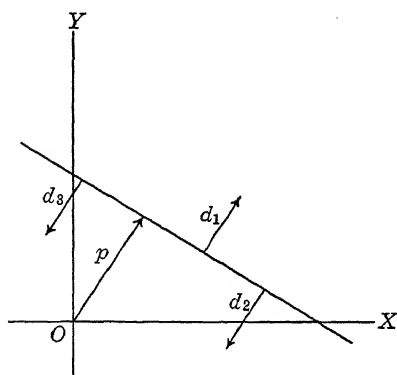


Figure 59.

Taking the direction of the normal as a standard, we shall consider a distance from a line to a point as positive if its direction is the same as that of the normal (see "Directed distances," Section 16); and we shall consider the distance negative

if its direction is opposite to that of the normal. Thus, in Figure 59,  $d_1$  is positive, and  $d_2$  and  $d_3$  are negative.

*Example 1*

Find the distance from the line  $4x - 3y + 4 = 0$  to the point  $(2, -1)$ .

Reducing the equation of the line to the normal form, we have:

$$\frac{4x - 3y + 4}{-5} = 0.$$

Hence, substituting,

$$d = \frac{4(2) - 3(-1) + 4}{-5} = -3.$$

The negative sign indicates that the direction of the distance from the line  $4x - 3y + 4 = 0$  to the point  $(2, -1)$  is opposite to the direction of the normal.

*Example 2*

Find the equations of the lines parallel to the line  $3x + 4y - 5 = 0$ , and passing at a distance of 3 units from it. (NOTE: This example may be treated as the following locus problem: "Find the equation of the locus of a point moving so that its distance from the line  $3x + 4y - 5 = 0$  always equals  $\pm 3$ .")

Since the distance of  $(x, y)$  from  $3x + 4y - 5 = 0$  equals  $\pm 3$ , hence we have:

$$(1) \frac{3x + 4y - 5}{5} = 3,$$

$$(2) \frac{3x + 4y - 5}{5} = -3.$$

Reducing the equations, we obtain the following two solutions:

$$(1) 3x + 4y - 20 = 0,$$

$$(2) 3x + 4y + 10 = 0.$$

*Example 3*

A point moves so that it is always equidistant from the lines  $3x + 4y - 12 = 0$  and  $5x - 12y - 10 = 0$ . Find the equation of the locus.

Since the distance of  $(x, y)$  from  $3x + 4y - 12 = 0$  equals  $\pm$  the distance of  $(x, y)$  from  $5x - 12y - 10 = 0$ , hence we have the two solutions;

$$(2) \frac{3x + 4y - 12}{5} = -\frac{5x - 12y - 10}{13}$$

Or, simplifying,

$$(1) 7x + 56y - 53 = 0,$$

$$(2) 32x - 4y - 103 = 0.$$

In Figure 60,  $CD$  is  $3x + 4y - 12 = 0$ , and  $AB$  is  $5x - 12y - 10 = 0$ .

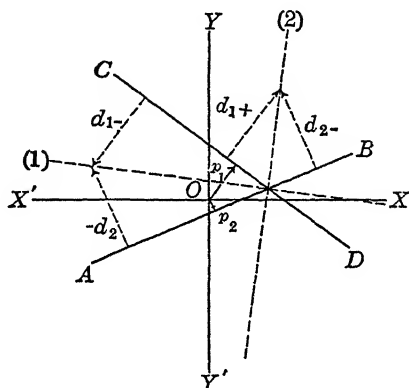


Figure 60.

Let

$$\frac{3x + 4y - 12}{5} = d_1,$$

and

$$\frac{5x - 12y - 10}{13} = d_2.$$

Then, line (1) is the locus obtained by letting  $d_1 = d_2$ . Its equation is:

$$7x + 56y - 53 = 0.$$

Likewise, line (2) is obtained by letting  $d_1 = -d_2$ . Its equation is:

$$32x - 4y - 103 = 0.$$

These results are evident from the figure since, for points on line (1),  $d_1$  and  $d_2$  have the same sign; and for points on line (2),  $d_1$  and  $d_2$  have opposite signs. The signs can be checked by com-

paring the directions of  $d_1$  and  $d_2$  with those of the normals  $p_1$  and  $p_2$ . Hence, if we wish to distinguish between lines (1) and (2), a fairly accurate figure is necessary.

From plane geometry, we know, also, that lines (1) and (2) are bisectors of the angles formed by the lines  $AB$  and  $CD$ , or  $3x + 4y - 12 = 0$  and  $5x - 12y - 10 = 0$ .

### Problems

1. Find the distance from  $3x - 4y - 10 = 0$  to each of the following points:  $(3, -1)$ ;  $(2, 0)$ ;  $(-1, 3)$ ;  $(4, 2)$ ; and  $(0, -\frac{5}{2})$ .

2. Find the distance from  $2x + y + 6 = 0$  to each of the following points:  $(1, 3)$ ;  $(-1, 2)$ ;  $(3, -1)$ ;  $(4, 0)$ ; and  $(0, -6)$ .

3. Find the equations of the lines parallel to  $2x - 3y - 6 = 0$ , and passing at a distance of 3 units from it.

4. Find the equations of the lines parallel to  $x - y + 3 = 0$ , and passing at a distance of 2 units from it.

5. Find the equations of the bisectors of the angles formed by the lines  $3x - 4y - 10 = 0$  and  $x + 2y - 3 = 0$ .

6. Find the equations of the bisectors of the angles formed by the lines  $x + y - 6 = 0$  and  $2x - y - 1 = 0$ .

7. A point moves so that its distance from the line  $x - 4y - 2 = 0$  always equals 5. Find the equation of the locus.

8. A point moves so that its distance from the line  $3x + 4y - 15 = 0$  always equals its distance from  $(-1, 2)$ . Find the equation of the locus.

9. Find the equation of the locus of a point moving so that its distance from  $(2, 4)$  always equals three times its distance from the line  $x - 2y + 3 = 0$ .

10. Find the equation of the locus of a point moving so that its distance from  $(1, -3)$  always equals one-half its distance from the line  $3x - 4y + 5 = 0$ .

11. The vertices of a triangle are:  $A(0, -2)$ ,  $B(4, 6)$ , and  $C(1, 4)$ . Find:

- The equation of the line joining  $B$  and  $C$ .
- The length of the side  $BC$ .
- The length of the altitude through  $A$ .
- The area of the triangle.

12. Find the area of a triangle with vertices at the following points:  $(2, -1)$ ,  $(3, 0)$ , and  $(-2, 4)$ .

13. The lines  $2x - y - 4 = 0$ ,  $x + y - 2 = 0$ , and  $x - 2y + 4 = 0$  form a triangle. Find the area.

14. A circle of radius 4 is tangent to the line  $2x - 3y + 2 = 0$ . Find the equation of the locus of its center.

15. Find the equations of the lines parallel to  $4x - 3y + 5 = 0$ , and twice as far as that line is from the origin.

16. Find the equations of the lines parallel to  $2x - 3y + 1 = 0$ , and 2 units farther from the origin.

17. Find the equations of the lines parallel to  $x + 3y - 6 = 0$ , and passing at a distance of 2 units from the point  $(-1, 2)$ .

18. Find the equations of the lines parallel to  $2x + 5y - 6 = 0$ , and passing at a distance of 3 units from  $(3, -4)$ .

19. The base of a triangle is the line joining  $(2, 4)$  and  $(-1, 0)$ . The area is 10 square units. Find the locus of the third vertex.

20. In Problem 19, if the triangle is isosceles, find the coördinates of the third vertex.

**60. Lines through the point of intersection of two given lines.** Suppose we are given the equations of two intersecting lines:

$$(1) A_1x + B_1y + C_1 = 0,$$

$$(2) A_2x + B_2y + C_2 = 0.$$

Let us multiply the second equation by  $k$ , and add it to the first equation. What does the resulting equation represent?

We wish to show that this equation

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0$$

represents for each value of  $k$ , a straight line through the intersection of the two given lines.

First, the equation is obviously that of a straight line, for the equation is of the first degree. Now, let us call the coördinates of the intersection point  $(x_1, y_1)$ . Then,

$$A_1x_1 + B_1y_1 + C_1 \equiv 0,*$$

---

\* " $\equiv 0$ " means equals zero identically, or vanishes identically—that is, all the terms cancel out.

for the point  $(x_1, y_1)$  lies on line (1). Also,

$$A_2x_1 + B_2y_1 + C_2 \equiv 0,$$

for the point  $(x_1, y_1)$  lies on line (2). Hence, substituting in the equation

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0,$$

we have:

$$0 + k \cdot 0 \equiv 0.$$

This solution proves that the point  $(x_1, y_1)$  lies on the line

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0,$$

for its coördinates satisfy the equation of that line.

### Example 1

Find the equation of the line through the point of intersection of the lines  $2x - y + 3 = 0$  and  $x + y - 1 = 0$ , and the point  $(1, -2)$ .

Since the required line passes through the point of intersection of  $2x - y + 3 = 0$  and  $x + y - 1 = 0$ , the line has an equation of the form

$$2x - y + 3 + k(x + y - 1) = 0.$$

Since the required line passes through  $(1, -2)$ , this equation must be satisfied by  $x = 1$ , and  $y = -2$ . Hence, we have:

$$2(1) + 2 + 3 + k(1 - 2 - 1) = 0,$$

or:

$$7 - 2k = 0.$$

Therefore:

$$k = \frac{7}{2}.$$

Thus, substituting the above value for  $k$ , we have the required equation:

$$2x - y + 3 + \frac{7}{2}(x + y - 1) = 0,$$

or:

$$4x - 2y + 6 + 7x + 7y - 7 = 0.$$

Therefore:

$$11x + 5y - 1 = 0.$$

*Example 2*

Find the equation of the line through the point of intersection of  $x - y - 6 = 0$  and  $2x + y - 3 = 0$ , and perpendicular to the line  $3x - 2y + 5 = 0$ .

The required line has the form

$$x - y - 6 + k(2x + y - 3) = 0.$$

Since this line is perpendicular to  $3x - 2y + 5 = 0$ , its slope  $m$  must be the negative reciprocal of the slope of  $3x - 2y + 5 = 0$ . Hence, its slope may be expressed:

$$m = -\frac{2}{3}$$

We next find the slope of the required line in terms of  $k$ . Rewriting, we have:

$$x(2k + 1) + y(k - 1) - 3k - 6 = 0.$$

Then, since  $\text{slope} = -\frac{2k + 1}{k - 1},$

hence:  $-\frac{2}{3} = -\frac{2k + 1}{k - 1},$

or:  $2k - 2 = 6k + 3.$

Therefore:  $k = -\frac{5}{4}.$

Hence, substituting for  $k$ , we have the required equation:

$$x - y - 6 - \frac{5}{4}(2x + y - 3) = 0,$$

or:  $2x + 3y + 3 = 0.$

**Problems**

1. Find the equation of the line passing through the intersection of  $x - 2y + 3 = 0$  and  $2x + 3y - 1 = 0$ , and through the point  $(2, -3)$ .

2. Find the equation of the line passing through the intersection of  $x + y - 2 = 0$  and  $3x - y + 6 = 0$ , and through the point  $(1, -2)$ .

3. Find the equation of the line passing through the intersection of  $x + 2y + 1 = 0$  and  $2x - y - 2 = 0$ , and parallel to  $3x - y + 2 = 0$ .

4. Find the equation of the line passing through the intersection of  $2x + y - 6 = 0$  and  $x - 2y + 1 = 0$ , and having a slope of  $\frac{3}{4}$ .

5. Find the equation of the line passing through the intersection of  $x - y + 2 = 0$  and  $2x + y - 1 = 0$ , and perpendicular to a line of slope  $-\frac{3}{5}$ .

6. Find the equation of the line passing through the intersection of  $x + y - 3 = 0$  and  $x - 2y - 4 = 0$ , and perpendicular to  $2x + 3y - 5 = 0$ .

7. Find the equation of the line passing through the intersection of  $2x + y - 1 = 0$  and  $x - 3y + 5 = 0$ , and having  $x$ -intercept 2.

8. Find the equation of the line passing through the intersection of  $3x - y - 6 = 0$  and  $x + y - 2 = 0$ , and making an angle of  $135^\circ$  with the  $x$ -axis.

9. Find the equation of the line passing through the intersection of  $x - 2y + 1 = 0$  and  $3x + y - 4 = 0$ , and parallel to the  $x$ -axis.

10. Find the equation of the line passing through the intersection of  $x + y - 2 = 0$  and  $2x - y - 4 = 0$ , and parallel to the  $y$ -axis.

11. Given two intersecting lines: (1)  $A_1x + B_1y + C_1 = 0$ , and (2)  $A_2x + B_2y + C_2 = 0$ . Find the value of  $k$  that will make

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0$$

represent: (a) line (1); and (b) line (2).

12. Given the lines  $A_1x + B_1y + C_1 = 0$  and  $A_2x + B_2y + C_2 = 0$ . What does

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0$$

represent if the lines are: (a) parallel? and (b) coincident?

### Miscellaneous Problems

1. Find the distance between the parallel lines  $2x - 4y - 3 = 0$  and  $4x - 8y + 1 = 0$ .

2. Find the equations of the lines parallel to  $4x - 3y - 2 = 0$ , and passing at a distance of 3 units from it.

3. The vertices of a triangle are:  $(1, 3)$ ,  $(2, -1)$ , and  $(0, 5)$ . Find:

- (a) The equations of the sides.
- (b) The lengths of the altitudes.
- (c) The angles of the triangle.
- (d) The area of the triangle.
- (e) The intersection of the medians.
- (f) The intersection of the perpendicular bisectors of the sides.
- (g) The intersection of the bisectors of the angles.
- (h) The intersection of the altitudes.

4. Using the triangle in Problem 3, show that the points of intersection of the medians, of the perpendicular bisectors of the sides, and of the altitudes all lie in the same straight line.

## CHAPTER XI

### THE CIRCLE

**61. Definition and equation of the circle.** A *circle* is defined as the locus of a point  $P$  moving so that its distance from a fixed point is constant. The fixed point is called the *center*; and the constant distance, the *radius*.

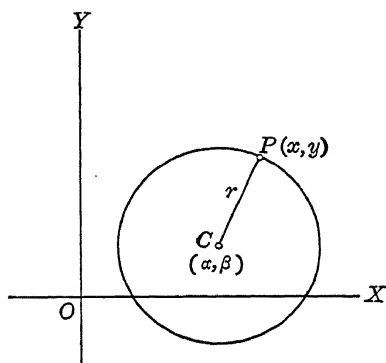


Figure 61.

The coördinates of the moving point  $P$  we shall call  $(x, y)$ ; the coördinates of the center,  $(\alpha, \beta)$ ; and the radius,  $r$ , as indicated in Figure 61. We desire the equation of the circle.

Since

$$\text{distance from } (\alpha, \beta) \text{ to } (x, y) = r,$$

hence:

$$\sqrt{(x - \alpha)^2 + (y - \beta)^2} = r,$$

or:

This equation is the *equation of the circle* with center at  $(\alpha, \beta)$  and with radius  $r$ .

It is evident that, if the center is at the origin, the equation will be

$$x^2 + y^2 = r^2.$$

*Example*

Find the center and the radius of the circle with the following equation:

$$x^2 + y^2 - 2x + 4y - 3 = 0.$$

We wish to reduce this equation to the form

$$(x - \alpha)^2 + (y - \beta)^2 = r^2.$$

Completing the square, we have:

$$x^2 - 2x + 1 + y^2 + 4y + 4 = 3 + 1 + 4,$$

or:  $(x - 1)^2 + (y + 2)^2 = 8.$

Hence we have:

$$\alpha = 1,$$

$$\beta = -2,$$

$$r = \sqrt{8};$$

or: center:  $(1, -2),$

radius:  $\sqrt{8}.$

### Problems

1. Find the center and the radius of each of the following circles, whose equations are:

(a)  $x^2 + y^2 - 4x + 2y - 3 = 0.$

(b)  $x^2 + y^2 + 6x - 8y - 5 = 0.$

(c)  $x^2 + y^2 + x - 2y - 1 = 0.$

(d)  $3x^2 + 3y^2 + 6x - 4y + 2 = 0.$

(e)  $5x^2 + 5y^2 - 6x - 2y + \frac{7}{5} = 0.$

2. Find the equations of the following circles: (a) with center at  $(2, -1)$ , and radius equal to 4; (b) with center at  $(-3, 4)$ , and radius equal to  $\sqrt{6}$ .

3. Find the equation of the circle with center at  $(2, -1)$ , and tangent to the  $y$ -axis.

4. Find the equation of the circle having its center at  $(-1, 4)$ , and passing through  $(3, 5)$ .

5. Find the equation of the circle that has as a diameter the line joining  $(3, -1)$  and  $(2, 5)$ .

6. Find the equation of the circle having its center at  $(-1, 3)$ , and passing through  $(2, 7)$ .

7. Find the equation of the circle with center at  $(5, 5)$ , and tangent to the  $x$ -axis.

8. Find the equation of the circle with center at  $(1, 3)$ , and tangent to the line  $3x - 4y - 10 = 0$ .

9. Find the equation of the circle with center at  $(1, -2)$ , and tangent to  $x + y - 6 = 0$ .

10. Find the equation of the circle with radius 5, and tangent to the line  $x - 2y + 6 = 0$  at  $(-4, 1)$ .

11. Prove by coordinates that an angle inscribed in a semi-circle is a right angle.

**62. General form of the equation of the circle.** In Section 57, we proved that a straight line is always represented by an equation of the first degree in  $x$  and  $y$ , and that the converse of this theorem is also true. We shall now discuss the corresponding theorem for the circle.

Expanding

$$(1) (x - \alpha)^2 + (y - \beta)^2 = r^2$$

and collecting terms, we have:

$$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0.$$

This equation may be written:

$$(2) x^2 + y^2 + Dx + Ey + F = 0,$$

where

$$D = -2\alpha,$$

$$E = -2\beta,$$

$$F = \alpha^2 + \beta^2 - r^2.$$

Moreover,  $D$ ,  $E$ , and  $F$  are constants but not necessarily integers. For example, the following is the equation of the circle with center at  $(\frac{1}{2}, -1)$  and with radius 2:

$$(x - \frac{1}{2})^2 + (y + 1)^2 = 4,$$

or:

$$x^2 + y^2 - x + 2y - \frac{17}{4} = 0.$$

Here

$$D = -1,$$

$$E = 2,$$

$$F = -\frac{17}{4}.$$

Since every circle has a center and a radius, and since a circle with a given center and a given radius has either the unique equation (1) or the same equation in a different

form (2), it follows that every circle is represented by an equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0$$

That is, every circle is represented by an equation of the *second* degree in  $x$  and  $y$ , with the coefficients of  $x^2$  and  $y^2$  unity and with no  $xy$  term.

Conversely, every equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0$$

represents a circle. For, completing the square, we have:

$$x^2 + Dx + \left(\frac{D}{2}\right)^2 + y^2 + Ey + \left(\frac{E}{2}\right)^2 = \left(\frac{D}{2}\right)^2 + \left(\frac{E}{2}\right)^2 - F,$$

or:

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4F}{4}.$$

This last equation is of the form

$$(x - \alpha)^2 + (y - \beta)^2 = r^2.$$

Observe that, if the result is to be a *real* circle, the quantity

$$\frac{D^2 + E^2 - 4F}{4} = r^2$$

must be positive. For, if the quantity is negative,  $r$  is imaginary—that is,  $r$  is the square root of a negative number; and if the quantity is zero,  $r$  is zero. Hence, for consistency, we say that the equation with imaginary  $r$  represents an *imaginary* circle; and that the equation with  $r$  equal to zero represents a *point* circle.

The following example illustrates the principles given above.

### Example

Find the equation of the circle passing through the three points (0, 2), (3, 3), and (−1, 1).

The required equation will be of the form

$$(2) \quad x^2 + y^2 + Dx + Ey + F = 0.$$

Hence, since the three points lie on the circle, their coördinates must satisfy the equation of the circle. Thus we have:

$$(3) \quad 0 + 4 + 0 \cdot D + 2E + F = 0$$

$$(4) \quad 9 + 9 + 3D + 3E + F = 0,$$

$$(5) \quad 1 + 1 - D + E + F = 0.$$

Solving these three equations for  $D$ ,  $E$ , and  $F$ , we obtain:

$$D = -6,$$

$$E = 4,$$

$$F = -12.$$

Therefore the required equation is:

$$x^2 + y^2 - 6x + 4y - 12 = 0.$$

There is another method of solution, which does not depend on form (2). This method is illustrated in the following:

### *Example*

We shall first find the center. From plane geometry, we know the center is the intersection of the perpendicular bisectors of the line segments joining any two pairs of points. By the method of Section 51, the equation of the perpendicular bisector of the line segment joining  $(0, 2)$  and  $(3, 3)$  is:

$$\sqrt{x^2 + (y - 2)^2} = \sqrt{(x - 3)^2 + (y - 3)^2},$$

or:

$$(6) \quad 3x + y - 7 = 0.$$

Similarly, the equation of the perpendicular bisector of the line segment joining  $(0, 2)$  and  $(-1, 1)$  is:

$$\sqrt{x^2 + (y - 2)^2} = \sqrt{(x + 1)^2 + (y - 1)^2},$$

or:

$$(7) \quad x + y - 1 = 0.$$

Solving (6) and (7), we find:

$$\begin{aligned}x &= 3, \\y &= -2.\end{aligned}$$

Hence the center is at the point  $(3, -2)$ . The distance from the center to any one of the three points is found to be 5. Therefore the required equation is:

$$(x - 3)^2 + (y + 2)^2 = 25,$$

or:

$$x^2 + y^2 - 6x + 4y - 12 = 0.$$

### Problems

1. Find the equations of the circles through the following points:

- (a)  $(2, 1)$ ,  $(-1, 3)$ ,  $(3, -2)$ .
- (b)  $(1, 1)$ ,  $(0, 6)$ ,  $(2, -3)$ .
- (c)  $(-2, 1)$ ,  $(1, -4)$ ,  $(3, -1)$ .

2. Find the equation of the circle having its center on the line  $x - 2y + 3 = 0$ , and passing through the points  $(1, 1)$  and  $(0, -3)$ .

3. Find the equation of the circle with center at  $(-3, 2)$ , and tangent to the line  $3x - 4y - 3 = 0$ .

4. Find the equation of the circle with radius 5, and tangent to the line  $2x - y + 4 = 0$  at  $(1, 6)$ .

5. Find the equation of the circle tangent to the line  $x + y - 2 = 0$  at  $(1, 1)$ , and passing through  $(2, 4)$ .

6. Find the equation of the circle tangent to  $2x + y - 4 = 0$  at  $(2, 0)$ , and passing through  $(3, -4)$ .

7. Find the equation of the circle passing through  $(2, 4)$  and  $(-1, 3)$  and having its center on the line  $x - 3y + 6 = 0$ .

8. Find the equation of the circle tangent to  $2x - y + 6 = 0$  and  $2x - y + 10 = 0$ , and having its center on the line  $x - 3y + 4 = 0$ .

9. Find the equation of the circle tangent to  $x + y - 3 = 0$  and  $x + y + 7 = 0$ , and having its center on the line  $2x + y - 4 = 0$ .

10. Find the equation of the circle with center on the  $x$ -axis, and tangent to the lines  $y = 4$  and  $x = 2$ .

11. Find the equation of the circle with center on the line  $y = 2$ , and tangent to the lines  $2x - 3y - 4 = 0$  and  $x - y - 6 = 0$ .

12. Find the equation of the circle that is tangent to the lines  $x = 0$ ,  $y = 0$ , and  $x = 5$ .

13. Find the equation of the circle inscribed in the triangle whose sides are the lines  $3x - 4y - 19 = 0$ ,  $4x + 3y - 17 = 0$ , and  $x - 7 = 0$ .

14. Find the equation of the circle inscribed in the triangle whose vertices are  $(0, 6)$ ,  $(8, 6)$ , and  $(0, 0)$ .

15. Find the equation of the circle which passes through  $(1, 7)$  and  $(8, 8)$ , and is tangent to the line  $3x + 4y - 6 = 0$ .

**63. Circles through the points of intersection of two given circles; radical axis.** In Section 60, we considered the equations of lines through the intersection point of two given lines. We shall apply the same treatment to the circle.

Given two circles:

$$(1) \quad x^2 + y^2 + A_1x + B_1y + C_1 = 0,$$

$$(2) \quad x^2 + y^2 + A_2x + B_2y + C_2 = 0.$$

We wish to show that the equation

$$(3) \quad x^2 + y^2 + A_1x + B_1y + C_1 + k(x^2 + y^2 + A_2x + B_2y + C_2) = 0$$

represents, for all values of  $k$  (with one exception), a circle through the intersection points (real or imaginary) of the two given circles.

Collecting terms, we have:

$$(4) \quad (x^2 + y^2)(1 + k) + x(A_1 + kA_2) + y(B_1 + kB_2) + C_1 + kC_2 = 0.$$

This equation is of the second degree and assumes, for all values of  $k$  except  $-1$ , the form of the circle equation (Section 62). We shall discuss the exception later. Furthermore, if  $(x_1, y_1)$  is an intersection point of the two circles, it lies on both circles, and equation (3) becomes:

$$0 + k \cdot 0 = 0.$$

Hence  $(x_1, y_1)$  lies on the circle represented by equation (3).

If the given circles are non-concentric but do not intersect in real points—that is, there are no *real* values  $(x, y)$  that satisfy the equations simultaneously—there will still be *imaginary* values that satisfy the equations simultaneously; hence, equation (3) will represent a circle through the *imaginary* points of intersection of the given circles.

If the given circles are concentric, there are no values  $(x, y)$ —real or imaginary—that satisfy the equations simultaneously; hence, equation (3) will represent a circle concentric with the given circles.

An interesting case arises when  $k$  equals  $-1$ . Equation (4) then becomes:

$$x(A_1 - A_2) + y(B_1 - B_2) + C_1 - C_2 = 0.$$

This is a first degree equation and, therefore, represents a straight line, which is called the *radical axis* of the two circles.

It is quite apparent that, if the equations of the circles are in the form of (1) and (2), the equation of the radical axis is obtained by subtracting one equation from the other. Thus, if (1) is represented by

$$S_1 = 0,$$

and (2), by

$$S_2 = 0,$$

the equation of the radical axis is:

$$S_1 - S_2 = 0,$$

or:

$$S_1 = S_2.$$

It is further apparent that, if the circles intersect in two different real points, the radical axis is the common chord.

Now, consider the three circles:  $S_1 = 0$ ,  $S_2 = 0$ , and  $S_3 = 0$ . We wish to show that the radical axes of the three circles (taken in pairs) meet in a common point called the *radical center*.

The equation of the radical axis of circles  $S_1 = 0$  and  $S_2 = 0$  is:

$$(5) \ S_1 - S_2 = 0.$$

The equation of the radical axis of circles  $S_2 = 0$  and  $S_3 = 0$  is:

$$(6) \ S_2 - S_3 = 0.$$

The equation of the radical axis of circles  $S_1 = 0$  and  $S_3 = 0$  is:

$$(7) \ S_1 - S_3 = 0.$$

However, if we add equations (5) and (6), we have, by Section 60, a line through the intersection of (5) and (6). Hence, adding (5) and (6), we obtain:

$$S_1 - S_3 = 0.$$

But this result is the same as equation (7).

Therefore, the line represented by (7)—that is,  $S_1 - S_3 = 0$ , which is the third radical axis—passes through the intersection of the other two. Or, in other words, the three radical axes meet in a common point.

### Problems

1. Find the equations of the following circles:

(a) Through the intersections of the circles  $x^2 + y^2 = 2x$  and  $x^2 + y^2 = 2y$ , and (3, -4).

(b) Through the intersections of  $x^2 + y^2 = 25$  and  $x^2 + y^2 - 2x + 4y - 6 = 0$ , and (1, 2).

(c) Through the intersections of the circle  $x^2 + y^2 = 16$  and the line  $x + y = 2$ , and (4, -3).

2. Find the radical axis of the circles  $x^2 + y^2 - 4x + 6y - 12 = 0$  and  $x^2 + y^2 + 2x - y + 3 = 0$ .

3. Find the radical axis of the circles  $x^2 + y^2 - x + y - 2 = 0$  and  $3x^2 + 3y^2 + 2x - 3y + 6 = 0$ .

4. Find the radical axis of the circles  $x^2 + y^2 - 6x - 4y + 9 = 0$  and  $x^2 + y^2 = 1$ .

5. Find the radical center of the circles  $x^2 + y^2 + 2x - y = 0$ ,  $x^2 + y^2 + x - y - 1 = 0$ , and  $x^2 + y^2 - 4x + 6y - 3 = 0$ .

6. Find the radical center of the circles  $x^2 + y^2 - x + 3y - 2 = 0$ ,  $x^2 + y^2 + 4x - 2y - 3 = 0$ , and  $x^2 + y^2 - 2x - 4y - 6 = 0$ .

7. In what case do two circles have no radical axis?

8. In what case do three circles have no radical center?

9. Give a geometric construction for the radical axis of two non-intersecting circles.

10. Prove analytically that the radical axis of two circles is perpendicular to their line of centers.

11. Prove analytically that the radical axis of two circles of equal radii is the perpendicular bisector of their line of centers.

12. What is the radical axis of two circles of zero radius?

13. Using the material in Problem 12, prove that the perpendicular bisectors of the sides of a triangle meet in a point.

14. If  $P(x_1, y_1)$  is a point outside the circle

$$(x - \alpha)^2 + (y - \beta)^2 - r^2 = 0,$$

show that the length of the tangent from  $P$  to the circle is given by the formula

$$t = \sqrt{(x_1 - \alpha)^2 + (y_1 - \beta)^2 - r^2}.$$

15. Using the result of Problem 14, show that the radical axis of two circles,  $S_1 = 0$  and  $S_2 = 0$ , may be defined as the locus of a point  $P$  moving so that the tangents drawn from  $P$  to the circles,  $S_1 = 0$  and  $S_2 = 0$ , are always equal.

## CHAPTER XII

### THE PARABOLA

**64. Definition of a conic.** A *conic* is defined as the locus of a point  $P$  moving so that its distance from a fixed point divided by its distance from a fixed line is a constant ratio. The fixed point is called the *focus*; and the fixed line, the *directrix*. The constant ratio is positive and is called the *eccentricity*. There are three types of conics:

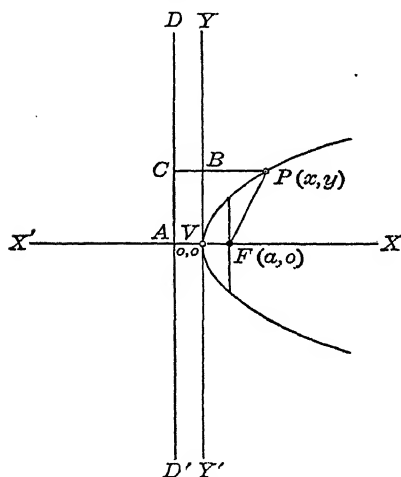


Figure 62.

(a) The parabola, when eccentricity equals 1.

(b) The ellipse, when eccentricity is less than 1.

(c) The hyperbola, when eccentricity is greater than 1.

We shall first consider the parabola.

**65. Definition and equation of the parabola.** A *parabola* is defined as the locus of a point  $P$  moving

so that its distance from the focus equals its distance from the directrix. We now wish the equation of a parabola.

Let us take the focus  $F$  on the  $x$ -axis, and the directrix  $DD'$  perpendicular to the  $x$ -axis at  $A$  (to the left of the focus), as in Figure 62. The point  $V$ —half-way between  $A$  and  $F$ , and equidistant from  $DD'$  and  $F$ —lies on the parabola. Let us call point  $V$  the *vertex*. Let us choose our origin of coördinates at the vertex, and let us call

$AF$ —that is, the distance from the directrix to the focus— $2a$ . Then,

$$AV = VF = a;$$

the coördinates of the focus are  $(a, o)$ ; the equation of the directrix is

$$x = -a;$$

and the coördinates of  $P$  are  $(x, y)$ . We may now proceed to the derivation of the equation.

Since

distance from  $(a, o)$  to  $(x, y)$  = distance from  $DD'$  to  $(x, y)$ ,  
hence:

$$PF = CP,$$

or:

$$\sqrt{(x - a)^2 + y^2} = a + x.$$

After squaring the above equation and collecting terms, we have:

$$y^2 = 4ax$$

This is the required *equation of the parabola*.

**66. Shape of the parabola.** To obtain an idea of the general shape of the parabola, let us examine the resulting equation

$$y = \pm \sqrt{4ax}.$$

When  $x$  is zero,  $y$  is zero (counted twice). Also, since we have chosen  $a$  as positive, it is evident that  $x$  must be zero or positive; for, if  $x$  were negative,  $y$  would be imaginary. Hence the curve passes through the origin where the curve itself touches the  $y$ -axis, and all other points on the curve lie to the right of the  $y$ -axis. Also, for every such value of  $x$ , there are two values of  $y$ , equal numerically but with opposite sign. Thus we say that the curve is *symmetrical with regard to the  $x$ -axis*. We call the line through the focus perpendicular to the directrix, the *axis*. Since  $x$  and  $y$  may assume as large values as we wish, the curve

may be said to extend indefinitely to the right of the  $y$ -axis, and to extend indefinitely above and below the  $x$ -axis. The general shape of the curve is given in Figure 63(a).

In the above discussion, since the directrix was taken to the left of the focus,  $a$  was considered positive. If, however, the directrix is taken to the right of the focus,  $a$  will

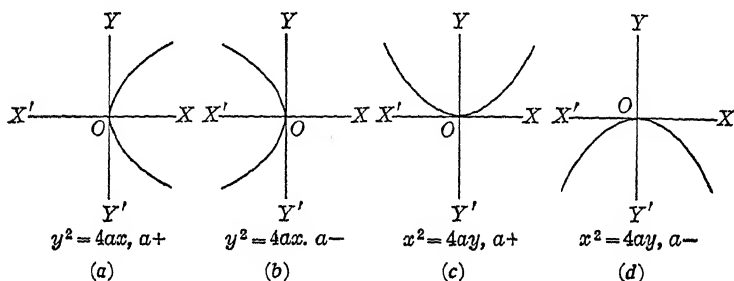


Figure 63.

be negative, and the parabola will *open to the left* instead of to the right. See Figure 63(b), on this page. The equation of the parabola will still be

the coördinates of the focus,  $(a, 0)$ ; and the equation of the directrix,

$$x = -a.$$

If we choose the focus on the  $y$ -axis, the directrix below the focus, and the origin of the coördinates at the vertex, the equation of the parabola will be:

$$x^2 = 4ay$$

and the parabola will *open upward*, as in Figure 63(c).

If the directrix is above the focus, the parabola will *open downward*, as in Figure 63(d).

The line through the focus perpendicular to the axis of the parabola and terminating on the parabola is called the

*latus rectum*. Its length equals  $4a$ . To illustrate, the coördinates of the ends of the latus rectum of the parabola

$$y^2 = 4ax$$

are  $(a, y)$ . Hence, substituting  $a$  for  $x$ , we have:

$$y^2 = 4a^2,$$

or:

$$y = \pm 2a.$$

Thus the total length of the latus rectum is equal to  $4a$ .

#### Example 1

Find the focus, the directrix, and the latus rectum of the parabola  $y^2 = 8x$ .

Since  $a$  is positive, this equation is in the form

$$y^2 = 4ax.$$

Then

$$4a = 8,$$

or

$$a = 2.$$

Hence we have the following:

focus:  $(2, 0)$

directrix: line  $(x = -2)$

latus rectum: 8

#### Example 2

Find the equation of the parabola with vertex at the origin and with focus the point  $(0, -3)$ .

Since the focus is on the  $y$ -axis, the equation is in the form

$$x^2 = 4ay.$$

Here

$$a = -3.$$

Hence the equation is:  $x^2 = -12y$ .

#### Problems

1. In each of the following parabolas, find the coördinates of the focus, the equation of the directrix, and the length of the latus rectum:

(a)  $y^2 = 8x$ .

(d)  $y^2 = -x$ .

(g)  $x^2 = 8y$ .

(b)  $y^2 = -16x$ .

(e)  $x^2 = -5y$ .

(h)  $x^2 = -16y$ .

(c)  $y^2 = 2x$ .

(f)  $4x^2 = 3y$ .

(i)  $x^2 = y$ .

2. In the following, find the equation of the parabola. The vertex in each case is at the origin.

- (a) Focus, the point  $(2, 0)$ .
- (b) Focus, the point  $(0, -4)$ .
- (c) Latus rectum 16; focus on the  $x$ -axis.
- (d) Axis, the  $y$ -axis; and passing through  $(-4, 2)$ .
- (e) Directrix, the line  $x = 5$ .
- (f) Latus rectum 8.

67. Equations of the parabola with vertex not at the origin. We have derived the equation of a parabola with its axis the  $x$ -axis and with its vertex at the origin, as

$$y^2 = 4ax.$$

We now wish to find the equation of a parabola with its axis parallel to the  $x$ -axis and with its vertex at the point  $(\alpha, \beta)$ . To do this, we must first derive a so-called *intrinsic property* of the parabola.

Consider any point  $P$  on the parabola  $y^2 = 4ax$ , as

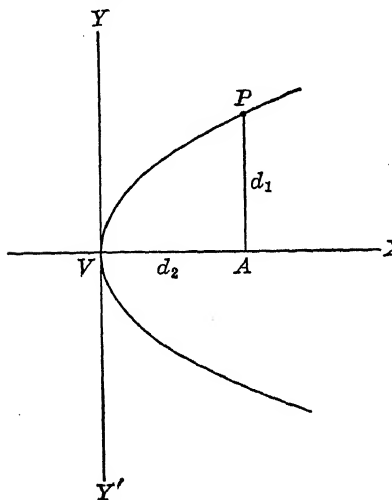


Figure 64.

indicated in Figure 64. Drop  $PA$  perpendicular to the axis of the parabola. Call this distance  $d_1$ , and the distance  $VA$  from the vertex  $V$  to the foot of the perpendicular,  $d_2$ . The coördinates of  $P$  are  $(d_2, d_1)$ . Since  $P$  lies on the parabola  $y^2 = 4ax$ , the coördinates of the point—that is,  $(d_2, d_1)$ —must satisfy the equation

$$y^2 = 4ax.$$

Therefore we have:

┌──────────┐

This equation is termed the *property* of the parabola.

(NOTE: This equation is true for any point on the parabola, and it is true regardless of the position of the parabola. The equation depends simply on the parabola itself—not on its position.)

Given the property equation

$$d_1^2 = 4ad_2,$$

we shall now proceed to find the equation of the parabola represented in Figure 65.

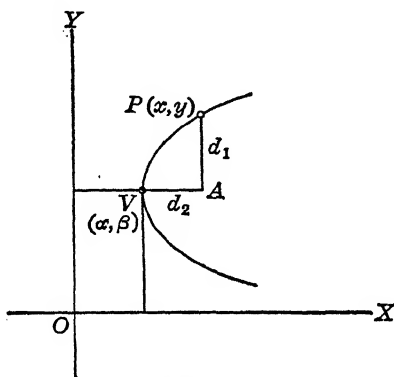


Figure 65.

Consider the parabola with its vertex  $V$  at  $(\alpha, \beta)$  and its axis parallel to  $OX$ . Take any point  $P(x, y)$  on the parabola, and drop a perpendicular to the axis. Then, since

$$d_1 = y -$$

and

$$d_2 = x - \alpha,$$

therefore, substituting, we have:

$$(1) \quad (y - \beta)^2 = 4a(x - \alpha).$$

This is the desired equation.

In like manner, if the axis is parallel to the  $y$ -axis, we shall have:

$$(3) \quad (x - \alpha)^2 = 4a(y - \beta).$$

The reader will observe that, for a parabola of the type

$$(y - \beta)^2 = 4a(x - \alpha),$$

we have the following results:

vertex:	$(\alpha, \beta)$
focus:	$(\alpha + a, \beta)$
directrix:	$x = \alpha - a$
axis:	$y = \beta$
latus rectum:	$4a$

Like results hold for the type

$$(3) \quad (x - \alpha)^2 = 4a(y - \beta).$$

Expanding the equation

$$(1) \quad (y - \beta)^2 = 4a(x - \alpha),$$

we have:

$$y^2 - 2\beta y + \beta^2 - 4ax + 4a\alpha = 0.$$

This equation may be written in the form:

$$(2) \quad y^2 + Ay + Bx + C = 0;$$

this is the form always assumed if the axis is parallel to  $OX$ . Similarly, the equation

$$(3) \quad (x - \alpha)^2 = 4a(y - \beta)$$

may be written

$$(4) \quad x^2 + Dx + Ey + F = 0.$$

The latter is the form always assumed if the axis is parallel to  $OY$ .

It will be quite evident, if we reverse the above steps, that every second degree equation with only one squared term and with no  $xy$  term may be reduced to form (1) or (3), and hence represents a parabola with its axis parallel to, or coincident with, one of the coördinate axes. We include in this category such an equation as

$$y^2 = 4.$$

This equation represents the two parallel lines  $y = 2$  and  $y = -2$ , but we call the result a *degenerate* form of parabola as the equation is of the parabola type.

*Example 1*

Given:  $y^2 + 2y - 4x + 9 = 0$ . Find vertex, focus, directrix, axis, and latus rectum.

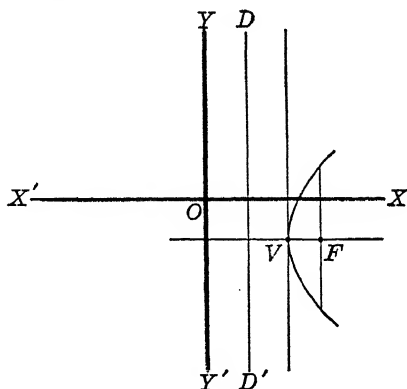


Figure 66.

We wish to reduce the equation to the form

$$(1) (y - \beta)^2 = 4a(x - \alpha).$$

Completing the square, we have:

$$y^2 + 2y + 1 = 4x - 9 + 1,$$

or:

$$(y + 1)^2 = 4(x - 2).$$

hence:

$$\alpha = 2,$$

$$\beta = -1.$$

Hence:

$$\text{vertex: } (2, -1)$$

$$4a = 4,$$

$$a = 1.$$

Therefore we have the following results:

$$\text{focus: } (3, -1)$$

$$\text{directrix: } x = +1$$

$$\text{latus rectum: } 4$$

$$\text{axis: } y = -1$$

The following procedure is a good check: Draw the figure for the above results. First, plot the vertex; then, indicate, along the axis to the focus, the distance  $a$  from the vertex; then, the distance  $-a$  from the vertex to the directrix.

### Example 2

Find the equation of the parabola with its vertex at  $(-2, 3)$  and its focus at  $(-2, 6)$ .

Since the  $x$ 's of the two points are the same, the parabola is of the type

$$(3) \quad (x - \alpha)^2 = 4a(y - \beta).$$

Since

$$a = 3,$$

hence:

$$(x + 2)^2 = 12(y - 3).$$

### Problems

1. In each of the following parabolas, find vertex, focus, directrix, axis, and latus rectum:

(a)  $y^2 - 4y - 4x = 0$ .

(b)  $y^2 + 2y + 8x - 15 = 0$ .

(c)  $x^2 - 4x - 3y + 6 = 0$ .

(d)  $x^2 - 2x + 2y - 5 = 0$ .

(e)  $4y^2 - 24y + x + 40 = 0$ .

(f)  $2x^2 - 8x - y + 12 = 0$ .

2. Find the equation of the parabola, given the following:

(a) Focus  $(3, -1)$ ; directrix,  $x = -5$ .

(b) Focus  $(2, 4)$ ; vertex  $(2, -2)$ .

(c) Vertex  $(3, 2)$ ; directrix,  $y = 7$ .

(d) Latus rectum 16; axis,  $y = 2$ ; vertex  $(5, 2)$ .

(e) Vertex  $(-1, 3)$ ; axis,  $y = 3$ ; passing through  $(3, 7)$ .

(f) Axis parallel to the  $x$ -axis; and passing through  $(0, 0)$ ,  $(2, -1)$ , and  $(2, 2)$ .

(g) Axis parallel to the  $y$ -axis; and passing through  $(2, 8)$ ,  $(1, -1)$ , and  $(-2, -4)$ .

## CHAPTER XIII

### THE ELLIPSE

**68. Definition and equation of the ellipse.** An *ellipse* is defined as the locus of a point  $P$  moving so that its distance from a fixed point divided by its distance from a fixed line is a constant less than 1. As in the case of the parabola, the fixed point is called the *focus*, and the fixed line, the *directrix*. The constant less than 1 we call the *eccentricity*, denoted by  $e$ .

In Figure 67, let us take the focus  $F$  on the  $x$ -axis, and the directrix  $DD'$  perpendicular to the  $x$ -axis at  $B$  and to the right of the focus. It is evident geometrically that the curve will cut the  $x$ -axis in two points,  $A'$  and  $A$ , called the *vertices*—one to the left of the focus, and the other between the focus and the directrix. Let us take  $O$ , the origin of the coördinates, midway between  $A'$  and  $A$ . Let  $a$  represent the distance  $OA$ . We wish to obtain expressions for  $OF$  and  $OB$  in terms of  $a$  and  $e$ .

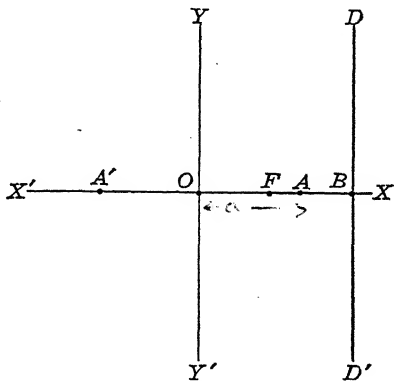


Figure 67.

By definition, we have the two relations:

$$(1) \frac{A'F}{A'B} = e,$$

$$(2) \frac{FA}{AB} = e;$$

or:

$$(3) A'F = e \cdot A'B,$$

$$(4) FA = e \cdot AB.$$

Simplifying to obtain  $a$ , we have:

$$(5) a + OF = e(a + OB),$$

$$(6) a - OF = e(OB - a).$$

Adding (5) and (6), we obtain:

$$2a = 2e \cdot OB,$$

or:

$$OB = \frac{a}{e}.$$

Subtracting, we have:

$$OF = ae.$$

Then, the coördinates of the focus are  $(ae, o)$ ; and the equation of the directrix is

$$x = \frac{a}{e}.$$

We may now proceed to find the equation of the ellipse.

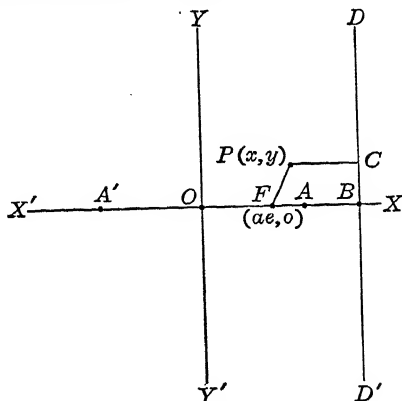


Figure 68.

Let  $P(x, y)$  represent any point on the ellipse. Then

$$\frac{\text{distance from } F \text{ to } P}{\text{distance from } DD' \text{ to } P} = e.$$

Or:

$$\frac{\sqrt{(x - ae)^2 + y^2}}{x - a/e} = e,$$

and:

$$\sqrt{(x - ae)^2 + y^2} = ex - a.$$

Squaring the equation and collecting terms, we have:

$$x^2(1 - e^2) + y^2 = a^2 - a^2e^2.$$

For convenience, we let

$$a^2 - a^2e^2 = b^2.$$

(This assumption is justifiable since  $e$  is less than 1.) Then, dividing by  $b^2$ , we have:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

This is the required *equation of the ellipse*.

**69. Shape of the ellipse.** Solving the above equation for  $y$ , we obtain:

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}.$$

Since for every value of  $x$  there are two values of  $y$ , equal numerically but with opposite sign, the ellipse is symmetrical with regard to the  $x$ -axis. If  $x$  is greater than  $a$ , or if  $x$  is less than  $-a$ ,  $y$  is imaginary; hence  $x$  may assume values (and all values) between and including  $-a$  and  $+a$ . If  $x$  equals  $\pm a$ ,  $y$  equals 0. We call  $A'A$  (as in Figure 69) the *major axis*; it is of length  $2a$  and always contains the focus.

Likewise, solving for  $x$ , we have:

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2}.$$

From this solution it follows that the ellipse is also symmetrical with regard to the  $y$ -axis; that  $y$  may assume all

values between and including  $+b$  and  $-b$ , and that  $OE$  equals  $E'O$  and equals  $b$ , as in Figure 69. From these conclusions it follows that the curve is *closed* and of the shape indicated. (NOTE: If  $a$  equals  $b$ , the ellipse becomes a circle.) We call  $E'E$  the *minor axis*; it is of length  $2b$ .

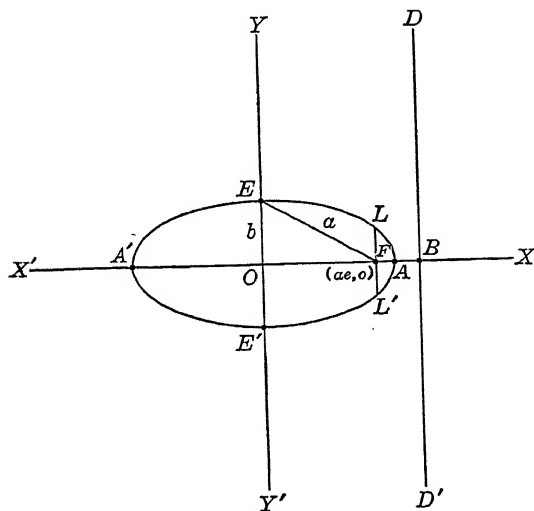


Figure 69.

The intersection of the major and the minor axes we call the *center* of the ellipse.

As in the parabola, the *latus rectum* is defined as the line through the focus perpendicular to the major axis and terminating on the ellipse. Its length is

$$\frac{2b^2}{a}.$$

To illustrate, the  $x$  coordinate of one end of the latus rectum is  $ae$ . Now, to find the  $y$  coordinate, we proceed as follows: For convenience we have taken the form indicated by

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}.$$

In the above equation of the ellipse, we substitute  $ae$  for  $x$ . Hence:

$$y = \pm \frac{b}{a} \sqrt{a^2 - a^2 e^2} = \pm \frac{b}{a} \sqrt{b^2} = \pm \frac{b^2}{a}.$$

Therefore the entire length of the latus rectum  $L'L$  is:

$$2 \frac{b^2}{a}.$$

It is easily apparent that the line joining the focus to one end of the minor axis is of length  $a$ .

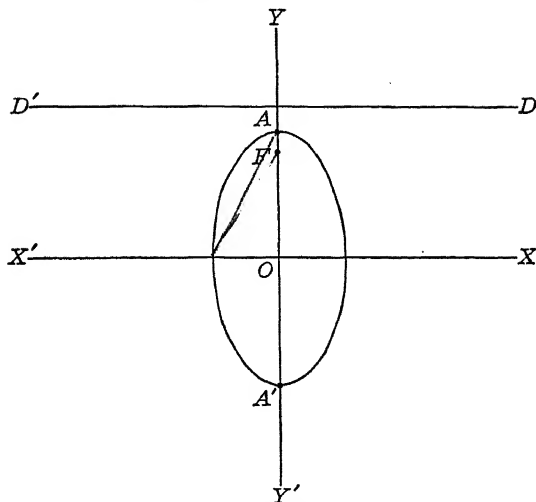


Figure 70.

Similarly, if, as in Figure 70, we choose the focus on the  $y$ -axis and the directrix perpendicular to the  $y$ -axis, and then proceed as above, our equation will be

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

The coördinates of the focus in this case are  $(0, ae)$ , and the equation of the directrix is

$$y = \frac{a}{e}.$$

The equation connecting the quantities  $a$ ,  $ae$ , and  $b$  is the same as in the previous case,

$$a^2 - a^2e^2 = b^2.$$

#### Example

Given an ellipse of the form

$$\frac{x^2}{25} + \frac{y^2}{16} = 1;$$

find focus, directrix, vertices, major and minor axes, latus rectum, and eccentricity.

This ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence:

$$a = 5,$$

$$b = 4.$$

From the relation

$$a^2e^2 = a^2 - b^2,$$

we have:

$$ae = 3.$$

Hence:

$$\frac{a}{e} = \frac{a^2}{ae} = \frac{25}{3},$$

$$e = \frac{ae}{a} = \frac{3}{5}.$$

Therefore we have the following results:

$$\text{focus:} \quad (3, 0)$$

$$\text{vertices:} \quad (\pm 5, 0)$$

$$\text{major axis:} \quad 10$$

$$\text{minor axis:} \quad 8$$

$$\text{directrix:} \quad x = \frac{25}{3}$$

$$\text{eccentricity:} \quad \frac{3}{5}$$

$$\text{latus rectum:} \quad \frac{32}{5}$$

### Problems

In the following, find focus, directrix, eccentricity, major axis, minor axis, latus rectum, and vertices.

$$1. \quad \frac{x^2}{16} + \frac{y^2}{9} = 1.$$

$$5. \quad \frac{x^2}{64} + \frac{y^2}{28} = 1.$$

$$2. \quad \frac{x^2}{9} + \frac{y^2}{2} = 1.$$

$$6. \quad x^2 + \frac{y^2}{2} = 1.$$

$$3. \quad \frac{x^2}{7} + \frac{y^2}{16} = 1.$$

$$7. \quad x^2 + 3y^2 = 4.$$

$$8. \quad 2x^2 + 5y^2 = 10.$$

$$9. \quad 3x^2 + y^2 = 6.$$

$$4. \quad \frac{x^2}{5} + \frac{y^2}{8} = 1.$$

$$10. \quad 2x^2 + 4y^2 = 1.$$

**70. Second focus and directrix.** In the discussion in Section 68, we chose the directrix to the right of the focus. It is evident that, if we choose the directrix to the left of the focus and choose the origin midway between  $A'$  and  $A$ , as before, the coördinates of the focus will be  $(-ae, o)$ , and the equation of the directrix,

$$x = -\frac{a}{e}.$$

Then, applying the definition of the ellipse, we have:

$$\frac{\sqrt{(x + ae)^2 + y^2}}{x + a/e} = e,$$

or, finally,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The above equation is the same one that we derived before, when the ellipse had its focus at  $(ae, o)$  and its directrix,

$$x = \frac{a}{e}.$$

From these results it follows that every ellipse of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has two foci with coördinates  $(\pm ae, o)$ , and two directrices with equations

$$x = \pm \frac{a}{e}.$$

Similar results follow for the ellipse of the form

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

### Problems

Find the equation of each of the following ellipses. In each instance the center is at the origin.

1. Foci at (3, 0) and (-3, 0); major axis 10.
2. Foci at (0, 4) and (0, -4); minor axis 6.
3. One focus at (2, 0); one directrix,  $x = -4$ .
4. Latus rectum  $\frac{16}{3}$ ; major axis 10; foci on  $x$ -axis.
5. Vertices at (0, 7) and (0, -7); minor axis 4.
6. One focus at (0, 3); one directrix,  $y = 7$ .
7. Distance between foci, 6; minor axis 8.
8. Foci on  $x$ -axis; distance between foci, 6; latus rectum  $\frac{7}{2}$ .

Answer:

$$\frac{x^2}{16} + \frac{y^2}{7} = 1.$$

9. Foci on  $y$ -axis; distance between foci, 4; eccentricity  $\frac{2}{3}$ .

Answer:

$$\frac{y^2}{9} + \frac{x^2}{5} = 1.$$

10. Foci on  $x$ -axis; eccentricity  $\frac{1}{2}$ ; latus rectum 6. Answer:

$$\frac{x^2}{16} + \frac{y^2}{12} = 1.$$

11. Foci on  $x$ -axis; passing through (2, 4) and (-4,  $2\sqrt{2}$ ).
12. Foci on  $y$ -axis; passing through (1, 4) and (3, 2).
13. Foci (4, 0) and (-4, 0); latus rectum 3.6.
14. Foci on  $y$ -axis; eccentricity  $\frac{3}{4}$ ; latus rectum  $\frac{7}{3}$ .
15. Directrices,  $x = \pm 4$ ; latus rectum 3.

**71. Equations of the ellipse with center not at the origin.** In Chapter XII we derived a so-called intrinsic property of the parabola and applied that property to find the equation of a parabola with its vertex at  $(\alpha, \beta)$ . We proceed similarly to find the equation of such an ellipse.

Consider the ellipse (Figure 71) of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

From any point  $P$ , drop perpendiculars  $AP$  and  $BP$  to the major and the minor axes, respectively. Let  $AP = d_1$ ,

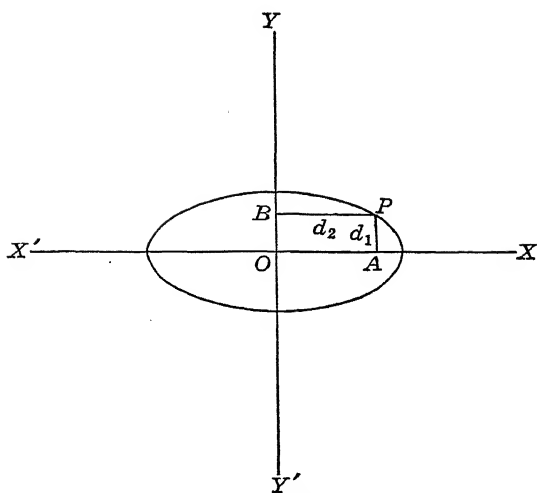


Figure 71.

and  $BP = d_2$ . Then the coördinates of  $P$  are  $(d_2, d_1)$ . Substituting in the equation of the ellipse, we have:

$$\frac{d_2^2}{a^2} + \frac{d_1^2}{b^2} = 1$$

This equation is the required *property* of the ellipse.

Let us now consider an ellipse with its center at  $(\alpha, \beta)$  and with its major axis parallel to the  $x$ -axis (Figure 72). Take any point  $P$  and drop perpendiculars to the major and the minor axes. Now, we already have the above property equation:

$$\frac{d_2^2}{a^2} + \frac{d_1^2}{b^2} = 1.$$

But

$$\begin{aligned} d_2 &= x - \alpha, \\ d_1 &= y - \beta. \end{aligned}$$

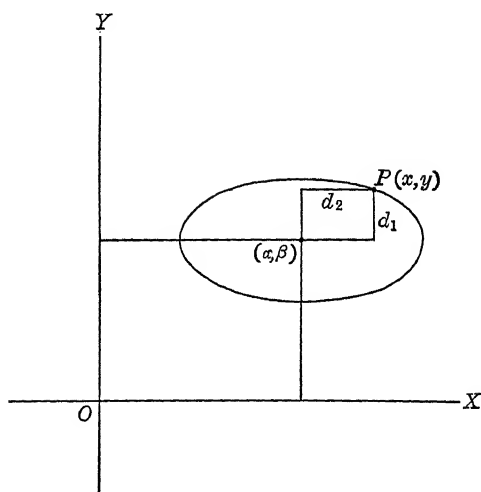


Figure 72.

Therefore :

$$(1) \quad \frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1$$

This is the required equation.

From this equation we observe :

center:  $(\alpha, \beta)$

foci:  $(\alpha \pm ae, \beta)$

vertices:  $(\alpha \pm a, \beta)$

directrices:  $x = \alpha \pm \frac{a}{e}$

Similar results obtain if we take an ellipse with its major axis parallel to the  $y$ -axis. The equation of the ellipse then is:

$$(2) \quad \frac{(y - \beta)^2}{a^2} + \frac{(x - \alpha)^2}{b^2} = 1$$

If we expand either (1) or (2), we shall have an equation of the form

$$(3) \quad Ax^2 + By^2 = 0.$$

Here,  $A$  and  $B$  have the *same* sign.

By reversing our steps and completing the square of (3), we may reduce this equation to either (2) or (1)—except that the right-hand side may possibly be 0 or  $-1$ .

If we include *point* ellipses and *imaginary* ellipses in our classification, then we can say that every equation of the form (3), that is, every second degree equation, involving both  $x^2$  and  $y^2$ , with coefficients of the same sign and no  $xy$  term—represents an ellipse with its major axis parallel to, or coincident with, one of the coördinate axes. Of course, if  $A$  equals  $B$ , the ellipse becomes a circle.

### Example

Find center, foci, directrices, vertices, axes, and latera recta of the following ellipse:

$$4x^2 + 9y^2 - 16x + 18y - 11 = 0.$$

Completing the square, we have:

$$4(x^2 - 4x + 4) + 9(y^2 + 2y + 1) = 11 + 16 + 9,$$

$$\text{or:} \quad 4(x - 2)^2 + 9(y + 1)^2 = 36,$$

$$\text{or:} \quad \frac{(x - 2)^2}{9} + \frac{(y + 1)^2}{4} = 1.$$

This last equation is of the form

$$\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1.$$

Hence we obtain the following:

$$\begin{aligned} a &= 3, \\ b &= 2, \\ ae &= \sqrt{a^2 - b^2} = \sqrt{5}; \\ \frac{a}{e} &= \frac{a^2}{ae} = \frac{9}{\sqrt{5}}; \end{aligned}$$

$$e = \frac{ae}{a} = \frac{\sqrt{5}}{3}.$$

Therefore we have these results:

$$\begin{aligned} \text{center:} & \quad (2, -1) \\ \text{foci:} & \quad (2 \pm \sqrt{5}, -1) \\ \text{vertices:} & \quad (2 \pm 3, -1) \\ \text{major axis:} & \quad 6 \\ \text{minor axis:} & \quad 4 \\ \text{eccentricity:} & \quad \frac{\sqrt{5}}{3} \\ \text{latera recta:} & \quad \frac{8}{3} \\ \text{directrices:} & \quad x = 2 \pm \frac{9}{\sqrt{5}} \end{aligned}$$

### Problems

1. Find center, foci, directrices, vertices, latera recta, axes, and eccentricity of the following:

$$(a) \frac{(x-2)^2}{25} + \frac{(y+1)^2}{9} = 1.$$

$$(b) \frac{(x+1)^2}{16} + \frac{(y-3)^2}{25} = 1.$$

$$(c) \frac{(x-3)^2}{16} + \frac{(y-2)^2}{7} = 1.$$

$$(d) \frac{(x+2)^2}{8} + \frac{(y-3)^2}{4} = 1.$$

$$(e) \frac{(x-4)^2}{169} + \frac{y^2}{144} = 1.$$

$$(f) 16x^2 + 25y^2 - 64x - 50y - 311 = 0.$$

$$(g) 3x^2 + 5y^2 + 6x - 30y + 33 = 0.$$

$$(h) 13x^2 + 4y^2 - 26x + 16y - 23 = 0.$$

$$(i) 3x^2 + 4y^2 - 16y + 15 = 0.$$

$$(j) x^2 + 2y^2 - 6x + 4y + 9 = 0.$$

$$(k) x^2 + y^2 - 2x + \quad - 1 = 0.$$

2. Find the equation of the ellipse with:

- (a) Foci at  $(-2, 3)$  and  $(4, 3)$ ; major axis 10.
- (b) Foci at  $(3, 5)$  and  $(3, 9)$ ; minor axis 8.
- (c) Foci at  $(2, 4)$  and  $(8, 4)$ ; equation of one directrix,  $x = 24$ .
- (d) Center at  $(5, -1)$ ; eccentricity  $\frac{3}{5}$ ; major axis 14.
- (e) Latus rectum  $\frac{8}{3}$ ; major axis 10; center at  $(3, 1)$ .
- (f) Center at  $(2, 4)$ ; one focus at  $(8, 4)$ ; eccentricity  $\frac{1}{2}$ .
- (g) Axes parallel to coördinate axes; and passing through  $(0, 0)$ ,  $(1, 2)$ ,  $(1, -1)$ , and  $(2, 1)$ .

3. Given the circle  $x^2 + y^2 = 64$ . From the circumference of this circle, perpendiculars are dropped to the  $x$ -axis. Find the equation of the locus of the mid-points of these perpendiculars.

4. Show that the distances to the foci from any point  $(x_1, y_1)$  on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

are:  $a + ex_1$ , and  $a - ex_1$ , and therefore that their sum is the constant  $2a$ .

5. From the result of Problem 4, form a new definition of the ellipse. Then, with this definition, show how an ellipse may be constructed by continuous motion. (NOTE: Use two thumb-tacks at the foci, and a piece of string of length  $2a$ .)

## CHAPTER XIV

### THE HYPERBOLA

**72. Definition and equation of the hyperbola.** An *hyperbola* is defined as the locus of a point moving so that its distance from a fixed point divided by its distance from a fixed line is a constant greater than 1. As in the ellipse, the fixed point is called the *focus*; the fixed line, the *directrix*; and the constant greater than 1, the *eccentricity*.

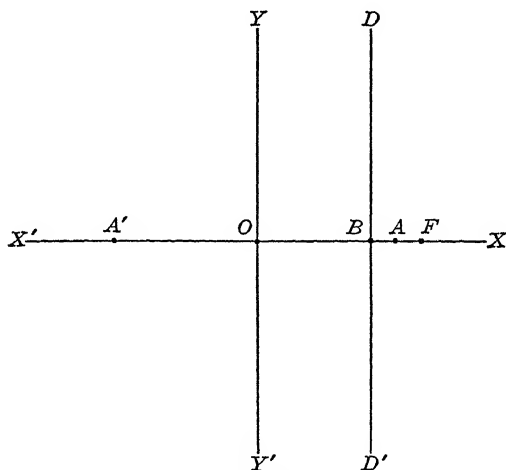


Figure 73.

We proceed, as with the ellipse, to take the focus on the  $x$ -axis, and the directrix perpendicular to the  $x$ -axis but now to the left of the focus (Figure 73). Points  $A'$  and  $A$  (the *vertices*) will exist as before, and we take the origin midway between them.

Now we have, as in the ellipse, the following relations:

$$OB = \frac{a}{e}$$

$$OF = ae$$

coördinates of focus:  $(ae, o)$

equation of directrix:  $x = \frac{a}{e}$

Hence we have also:

$$\frac{\sqrt{(x - ae)^2 + y^2}}{x - a/e} = e,$$

and, finally,

$$x^2(1 - e^2) + y^2 = a^2 - a^2e^2.$$

But now, since  $e$  is greater than 1, the quantity  $(a^2 - a^2e^2)$  is negative. Hence, we let

$$a^2 - a^2e^2 = -b^2,$$

and we have:

$$a^2e^2 = a^2 + b^2.$$

This equation is the relation between  $ae$  and  $a$  and  $b$  for the hyperbola. The final reduction gives us the *equation of the hyperbola*:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

**73. Shape of the hyperbola.** Solving the above equation for  $y$ , we obtain:

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$$

As in the case of the ellipse, the hyperbola is symmetrical with regard to the  $x$ -axis (Figure 74). If  $x$  is less than  $a$  but greater than  $-a$ ,  $y$  is imaginary. When  $x$  equals  $\pm a$ ,  $y$  equals 0; hence,  $x$  may assume values (and all values) only greater than or equal to  $a$ , or less than or equal to  $-a$ . This conclusion proves that there are no points of the curve

between  $(-a, 0)$  and  $(a, 0)$ . We call  $A'A$ , as in Figure 74, the *transverse axis*. It is of length  $2a$  and, extended, contains the focus. As  $x$  increases positively,  $y$  increases both positively and negatively. Likewise, as  $x$  increases negatively,  $y$  increases both positively and negatively. Hence, the hyperbola extends indefinitely to the right of the  $y$ -axis and above and below the  $x$ -axis; and it also extends indefinitely to the left of the  $y$ -axis and above and below the  $x$ -axis.

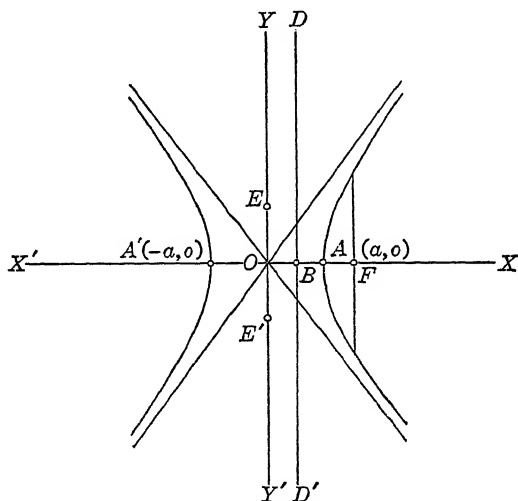


Figure 74.

Similarly, solving for  $x$ , we obtain:

$$x = \pm \sqrt{a^2 + y^2}.$$

From this equation it follows that the hyperbola is also symmetrical with regard to the  $y$ -axis, and that  $y$  may assume all values from  $-\infty$  to  $+\infty$ . The curve is of the shape indicated in Figure 74.

In order to preserve symmetry, we call the line  $E'E$  (Figure 74), joining  $(0, -b)$  and  $(0, b)$ , the *conjugate axis*.

Its length is  $2b$ . The intersection of the transverse and the conjugate axes we call the *center* of the hyperbola.

As in the ellipse, the latus rectum, perpendicular to the transverse axis at the focus, is of length

$$\frac{2b^2}{a}.$$

It is evident that the line joining  $(a, 0)$  and  $(0, b)$  is of length  $ae$ .

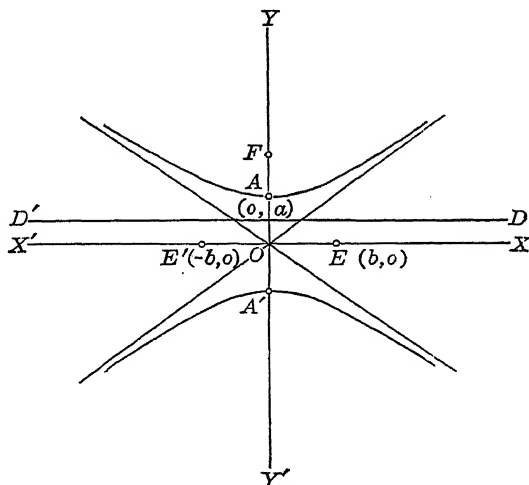


Figure 75.

In a similar manner, if we choose the focus on the  $y$ -axis and the directrix perpendicular to the  $y$ -axis (as in Figure 75), and proceed as above, our equation will be:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

The coördinates of the focus are  $(0, ae)$ , and the equation of the directrix is

$$y = \frac{a}{e}.$$

*Example*

Find focus, directrix, vertices, eccentricity, latus rectum, transverse axis, and conjugate axis of the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{16} = 1.$$

This is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

(NOTE: In the hyperbola,  $a$  may be less than  $b$  or equal to  $b$  or greater than  $b$ .)

Hence:

$$a^2 = 9,$$

$$b^2 = 16,$$

$$c^2 e^2 = a^2 + b^2 = 25;$$

or:

$$ae = 5,$$

$$\frac{a}{e} = \frac{a^2}{ae} = \frac{9}{5},$$

$$e = \frac{ae}{a} = \frac{5}{3}.$$

Hence we have the following results:

$$\begin{array}{ll} \text{focus:} & (5, 0) \\ \text{transverse axis:} & 6 \\ \text{conjugate axis:} & 8 \\ \text{vertices:} & (\pm 3, 0) \\ \text{latus rectum:} & \frac{32}{3} \\ \text{directrix:} & x = \frac{9}{5} \\ \text{eccentricity:} & \frac{5}{3} \end{array}$$

**Problems**

Find focus, directrix, eccentricity, vertices, latus rectum, transverse axis, and conjugate axis of:

- |  |   |
|--|---|
| 1. $x^2 - y^2 = 6.$                      | 6. $y^2 - x^2 = 1.$                       |
| 2. $x^2 - y^2 = -6.$                     | 7. $2x^2 - 3y^2 = 9.$                     |
| 3. $x^2 - 2y^2 = 8.$                     | 8. $x^2 - 3y^2 = 9.$                      |
| 4. $\frac{x^2}{3} - \frac{y^2}{6} = 1.$  | 9. $\frac{y^2}{25} - \frac{x^2}{24} = 1.$ |
| 5. $\frac{x^2}{16} - \frac{y^2}{9} = 1.$ | 10. $\frac{y^2}{2} - \frac{x^2}{7} = 1.$  |

**74. Second focus and directrix.** As in the case of the ellipse, it is evident that every hyperbola of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has two foci with coördinates  $(\pm ae, o)$ , and two directrices with equations

$$x = \pm \frac{a}{e}.$$

Similar results follow for the hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

**75. Asymptotes.** With every hyperbola there are associated two lines of interest and importance called *asymptotes*. We shall derive the equations of these lines.

Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in the form

$$\frac{y}{x} = \pm \frac{b}{a} \sqrt{1 - \frac{a^2}{x^2}}.$$

Let  $x$  increase indefinitely. As it does, the quantity

$$\pm \frac{b}{a} \sqrt{1 - \frac{a^2}{x^2}}$$

approaches

$$\pm \frac{b}{a} \cdot 1.$$

Thus

$$\frac{y}{x}$$

approaches

$$\pm \frac{b}{a}$$

Hence, as  $x$  increases indefinitely, the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

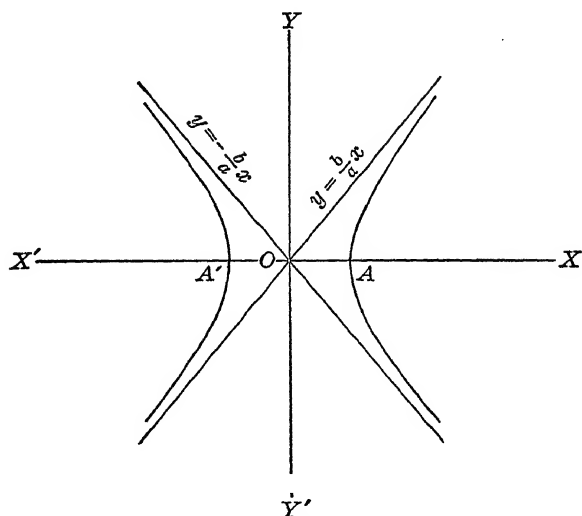


Figure 76.

approaches the two straight lines

$$\frac{y}{x} = \pm \frac{b}{a},$$

or, in simpler form,

$$y = \pm \frac{b}{a}x.$$

We call these lines (Figure 76) the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The equations of the asymptotes may be written:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Similarly, the equations of the asymptotes of the hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

will be:

$$y = \pm \frac{a}{b}x$$

or

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 0.$$

### Problems

1. In each of the following, find the equation of the hyperbola. In each case, the center is at the origin.

- (a) One focus at (5, 0); one vertex at (4, 0).
- (b) One directrix,  $x = 2$ ; one focus at  $(-4, 0)$ .
- (c) Transverse axis 8; conjugate axis 10.
- (d) One directrix,  $x = \frac{5}{3}$ ; transverse axis 20.
- (e) One directrix,  $y = \frac{1}{5}x$ ; conjugate axis 6.
- (f) Foci on  $x$ -axis; distance between foci 8; latus rectum  $\frac{1}{3}$ .
- (g) Foci  $(0, \pm 6)$ ; eccentricity 2.
- (h) Directrices parallel to  $y$ -axis; distance between directrices,  $\frac{8}{3}$ ; distance between foci, 6.
- (i) Latus rectum 4; slope of asymptotes,  $\pm \frac{1}{3}$ .
- (j) Foci on  $y$ -axis; passing through  $(-1, 1)$  and  $(5, -3)$ .
- (k) One directrix,  $x = \frac{1}{2}$ ; latus rectum 6.
- (l) Eccentricity  $\sqrt{2}$ ; slope of asymptotes,  $\pm 1$ .

(m) Foci on  $x$ -axis; passing through  $(3, 2)$  and  $(-\sqrt{6}, 0)$ .

(n) Asymptotes,  $\frac{x^2}{4} - \frac{y^2}{2} = 0$ ; foci on  $y$ -axis; conjugate axis 6.

2. Prove analytically that an hyperbola cannot intersect its asymptotes.

**76. Equations of the hyperbola with center not at the origin.** Exactly as in the case of the ellipse, we shall proceed to find the equation of the hyperbola with its center at  $(\alpha, \beta)$  and with its transverse axis parallel to the  $x$ -axis.

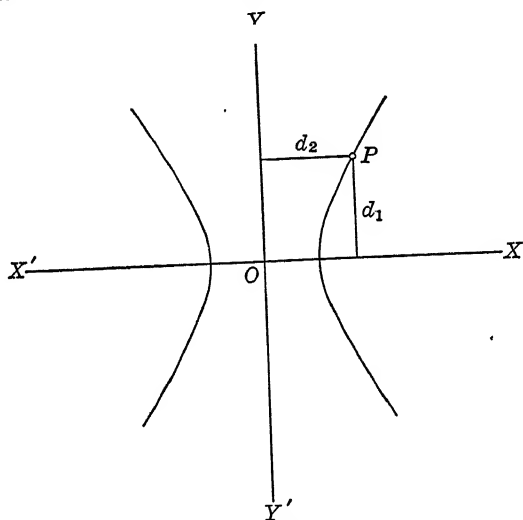


Figure 77.

Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

From any point  $P$ , drop perpendiculars to the transverse and the conjugate axes. Call them  $d_1$  and  $d_2$ , respectively. Then the coördinates of  $P$  are  $(d_2, d_1)$ . Substituting, we derive the *property* of the hyperbola:

$$\frac{d_2^2}{a^2} - \frac{d_1^2}{b^2} = 1$$

Now, consider the hyperbola with its center at  $(\alpha, \beta)$  and with its transverse axis parallel to the  $x$ -axis. From any point  $P$ , drop perpendiculars to the transverse and the conjugate axes.

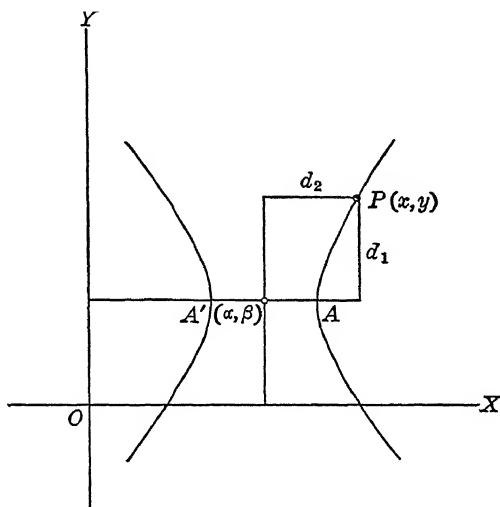


Figure 78.

We already have the property equation:

$$\frac{d_2^2}{a^2} - \frac{d_1^2}{b^2} = 1.$$

But

$$d_2 = x - \alpha,$$

$$d_1 = y - \beta.$$

Hence:

$$(1) \quad \boxed{\frac{(x - \alpha)^2}{a^2} - \frac{(y - \beta)^2}{b^2} = 1}$$

This is the required *equation of the hyperbola*.

From this equation we observe:

center:  $(\alpha, \beta)$

foci:  $(\alpha \pm ae, \beta)$

vertices:  $(\alpha \pm a, \beta)$

$$\text{directrices: } x = \alpha \pm \frac{a}{e}$$

$$\text{asymptotes: } \frac{(x - \alpha)^2}{a^2} - \frac{(y - \beta)^2}{b^2} = 0$$

Similar results obtain if we take the transverse axis parallel to the  $y$ -axis. The equation of the hyperbola then is:

$$(2) \quad \frac{(y - \beta)^2}{a^2} - \frac{(x - \alpha)^2}{b^2} = 1$$

If we expand either (1) or (2), we obtain an equation of the form

$$(3) \quad Ax^2 + By^2 + Cx + Dy + E = 0.$$

Here,  $A$  and  $B$  have *opposite* signs.

By reversing our steps, we may reduce equation (3) to either (2) or (1)—except that the right-hand side may possibly be 0. If the side equals 0, equation (3) represents two intersecting lines, or a *degenerate* hyperbola.

Including this case, we may say, then, that every equation of the form (3)—that is, every second degree equation, including both squared terms with *opposite* signs and no  $xy$  term—represents an hyperbola with its transverse axis parallel to, or coincident with, one of the coördinate axes.

### Example 1

Given an hyperbola of the form

$$3x^2 - 5y^2 + 6x + 20y - 32 = 0.$$

Find center, focus, directrix, eccentricity, vertices, transverse axis, conjugate axis, latus rectum, and asymptotes.

Completing the square, we have:

$$3(x^2 + 2x + 1) - 5(y^2 - 4y + 4) = 32 + 3 - 20,$$

$$\text{or: } 3(x + 1)^2 - 5(y - 2)^2 = 15,$$

$$\text{or: } \frac{(x + 1)^2}{5} - \frac{(y - 2)^2}{3} = 1.$$

This equation is in the form

$$\frac{(x - \alpha)^2}{a^2} - \frac{(y - \beta)^2}{b^2} = 1.$$

Hence we have the following:

$$\begin{aligned} a^2 &= 5, \\ b^2 &= 3, \\ a^2 e^2 &= a^2 + b^2 = 5 + 3 = 8; \\ \frac{a}{e} &= \frac{a^2}{ae} = \frac{5}{\sqrt{8}}; \\ e &= \frac{ae}{a} = \frac{\sqrt{8}}{\sqrt{5}} = \sqrt{\frac{8}{5}}. \end{aligned}$$

Therefore we have these results:

$$\begin{aligned} \text{center:} & \quad (-1, 2) \\ \text{foci:} & \quad (-1 \pm \sqrt{8}, 2) \\ \text{vertices:} & \quad (-1 \pm \sqrt{5}, 2) \\ \text{transverse axis:} & \quad 2\sqrt{5} \\ \text{conjugate axis:} & \quad 2\sqrt{3} \\ \text{directrices:} & \quad x = -1 \pm \frac{5}{\sqrt{8}} \\ \text{eccentricity:} & \quad \sqrt{\frac{8}{5}} \\ \text{latus rectum:} & \quad \frac{6}{\sqrt{5}} \\ \text{asymptotes:} & \quad \frac{(x + 1)^2}{5} - \frac{(y - 2)^2}{3} = 0 \end{aligned}$$

### Example 2

Find the equation of the hyperbola with foci at (1, 2) and (7, 2), and with conjugate axis 4.

Since the foci are on the line  $y = 2$ , the equation will be of the form

$$\frac{(x - \alpha)^2}{a^2} - \frac{(y - \beta)^2}{b^2} = 1.$$

Hence: 
$$\alpha = \frac{1+7}{2} = 4,$$

$$\beta = 2.$$

Since 
$$2b = 4,$$

$$b = 2;$$

and since 
$$ae = 3,$$

therefore: 
$$a^2 = a^2e^2 - b^2 = 9 - 4 = 5.$$

Hence: 
$$\frac{(x-4)^2}{5} - \frac{(y-2)^2}{4} = 1.$$

### Problems

1. Find center, foci, directrices, eccentricity, vertices, transverse and conjugate axes, latera recta, and asymptotes of the following:

(a) 
$$\frac{(x-1)^2}{9} - \frac{(y+3)^2}{16} = 1.$$

(b) 
$$\frac{(x-2)^2}{9} - \frac{(y+3)^2}{16} = -1$$

(c) 
$$\frac{(x-2)^2}{8} - \frac{(y-1)^2}{1} = 1.$$

(d) 
$$\frac{(y-3)^2}{7} - \frac{(x+2)^2}{9} = 1.$$

(e) 
$$\frac{(x-3)^2}{8} - \frac{(y-1)^2}{6} = 0.$$

(f) 
$$\frac{(x+1)^2}{3} - \frac{(y-2)^2}{1} = 1.$$

(g) 
$$\frac{(x-3)^2}{4} - \frac{(y-2)^2}{5} = 1.$$

(h) 
$$(x-1)^2 - (y-2)^2 = 1.$$

(i) 
$$9x^2 - 16y^2 - 54x - 64y - 127 = 0.$$

(j) 
$$6x^2 - 9y^2 - 48x + 18y + 141 = 0.$$

(k) 
$$20x^2 - 16y^2 - 40x - 32y + 324 = 0.$$

(l) 
$$4x^2 - 5y^2 - 16x + 10y - 9 = 0.$$

(m) 
$$x^2 - y^2 + 2x + 6y - 16 = 0.$$

(n) 
$$x^2 - y^2 + 2x = 0.$$

2. Find the equation of the hyperbola with:

- (a) Foci at  $(3, -1)$  and  $(17, -1)$ ; transverse axis 10.
- (b) Foci at  $(2, 2)$  and  $(2, -8)$ ; conjugate axis 8.
- (c) Center at  $(2, 4)$ ; one focus at  $(10, 4)$ ; one directrix,  $x = 4$ .
- (d) Vertices at  $(8, 6)$  and  $(8, -4)$ ; latus rectum  $\frac{16}{3}$ .
- (e) Eccentricity 3; center at  $(2, -3)$ ; one focus at  $(8, -3)$ .
- (f) Transverse axis 16; foci on line  $y = -1$ ; asymptotes:

$$\frac{(x-2)^2}{16} - \frac{(y+1)^2}{8} = 0.$$

3. Find the equation of the hyperbola with axes parallel to the coördinate axes, and passing through  $(1, 0)$ ,  $(0, 2)$ ,  $(-1, 2)$ , and  $(3, -1)$ .

4. Show that the distances to the foci from any point  $(x_1, y_1)$  on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are:  $ex_1 + a$ , and  $ex_1 - a$ , and that therefore their difference is the constant  $2a$ .

5. From the result of Problem 4, form a new definition of the hyperbola. Then, with this definition, show how an hyperbola may be constructed by continuous motion. (NOTE: Use two thumbtacks at the foci, and a piece of string.)

6. Two hyperbolas so related that the transverse axis of each is the conjugate axis of the other are called *conjugate* hyperbolas. Find the equation of the hyperbola conjugate to

$$\frac{x^2}{9} - \frac{y^2}{7} = 1.$$

7. Prove that two conjugate hyperbolas have the same asymptotes.

8. If  $e$  and  $e'$  are the eccentricities of two conjugate hyperbolas, prove:

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1.$$

9. Show that the foci of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and those of its conjugate all lie on the circle whose equation is:  $x^2 + y^2 = a^2 + b^2$ .

10. Show that the circle of Problem 9 meets either of the two hyperbolas on the directrix of the other.

11. Show that the line joining the focus of an hyperbola to the focus of its conjugate, passes through the point of intersection of the directrices of the two hyperbolas.

12. An hyperbola with  $a$  equal to  $b$  is called *equilateral*. Find the eccentricity of such an hyperbola.

13. A latus rectum of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is extended by the amount  $k$  so that it just reaches an asymptote. Show that  $k$  is equal to the radius of a circle inscribed in the triangle formed by the asymptotes and the line  $x = a$ .

## CHAPTER XV

### TRANSFORMATION OF COÖRDINATES

**77. Translation of axes.** In Chapter XI while discussing the equations of circles, we saw that the equation of a circle with its center at the origin was

$$x^2 + y^2 = r^2;$$

whereas, for the circle with its center at  $(\alpha, \beta)$ , the equation was

$$(x - \alpha)^2 + (y - \beta)^2 = r^2.$$

It is quite evident that the equation of the circle with its center at the origin assumes a simpler form than that assumed when the center is not at the origin. Similar conclusions obtain for the ellipse and the hyperbola, and also for the parabola if we consider the vertex in place of a center.

Therefore, given any curve with the center or some other element not at the origin, it would be quite convenient if we could *move the coördinate axes* in such a way that the origin would coincide with this point, and the coördinate axes would coincide with the axes of the curve, if such axes existed. This procedure is possible. There are two motions involved: *translation*, or moving the coördinate axes parallel to themselves; and *rotation*, or turning the axes about a point.

We shall first consider translation, which involves changing the origin without changing the directions of the axes.

We are given two sets of axes (Figure 79): the original set,  $OX$  and  $OY$ , with origin at  $O(o, o)$ ; and a new set,  $CX'$  and  $CY'$ , with origin at  $C(h, k)$ . We are given, also, a point  $P$ , whose coördinates referred to the original axes

are  $x = OA$  and  $y = AP$ , and whose coördinates referred to the new axes are  $x' = CB$  and  $y' = BP$ . We wish to find the relations between the primed and unprimed quantities.

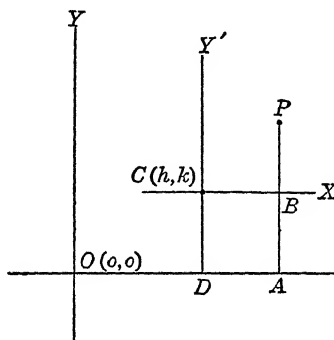


Figure 79.

We have:

$$OA = OD + DA,$$

or:

$$x = h + x'.$$

Also:

$$AP = AB + BP,$$

or:

$$y = k + y'.$$

Hence we have the *equations of translation*:

$\begin{aligned} x &= x' + h \\ y &= y' + k \end{aligned}$
--

It is quite evident, then, if we are given the equation of a curve referred to any set of axes, that the equation of the same curve referred to new axes—parallel to the given axes and with origin at  $(h, k)$ —is obtained by replacing  $x$  with  $x' + h$  and  $y$  with  $y' + k$ .

#### Example 1

Find the equation of the circle

$$x^2 + y^2 - 6x + 2y - 6 = 0$$

referred to parallel axes through (3, -1).

$$\begin{aligned}\text{Since} \quad & x = x' + 3 \\ \text{and} \quad & y = y' - 1,\end{aligned}$$

we substitute in the circle equation and obtain:

$$x'^2 + 6x' + 9 + y'^2 - 2y' + 1 - 6x' - 18 + 2y' - 2 - 6 = 0.$$

After simplifying this result, we have:

$$x'^2 + y'^2 = 16.$$

### *Example 2*

Remove the first degree terms from the following:

$$x^2 + y^2 - 2x + 4y - 4 = 0.$$

$$\begin{aligned}\text{Since} \quad & x = x' + h \\ \text{and} \quad & y = y' + k,\end{aligned}$$

then, substituting in the above equation, we obtain:

$$\begin{aligned}x'^2 + 2hx' + h^2 + y'^2 + 2ky' + k^2 \\ - 2x' - 2h + 4y' + 4k - 4 = 0.\end{aligned}$$

(NOTE: In order for the first degree terms to vanish, their coefficients must equal zero.)

Collecting the first degree terms, we have:

$$\begin{aligned}x'(2h - 2), \\ y'(2k + 4).\end{aligned}$$

$$\begin{aligned}\text{Hence:} \quad & 2h - 2 = 0, \\ & 2k + 4 = 0;\end{aligned}$$

$$\begin{aligned}\text{or:} \quad & h = 1, \\ & k = -2.\end{aligned}$$

Substituting again, we have:

$$x'^2 + y'^2 + 1 + 4 - 2 - 8 - 4 = 0,$$

$$\text{or, finally,} \quad x'^2 + y'^2 = 9.$$

We might have solved Example 2 by another procedure, indicated in the succeeding text:

are  $x = OA$  and  $y = AP$ , and whose coördinates referred to the new axes are  $x' = CB$  and  $y' = BP$ . We wish to find the relations between the primed and unprimed quantities.

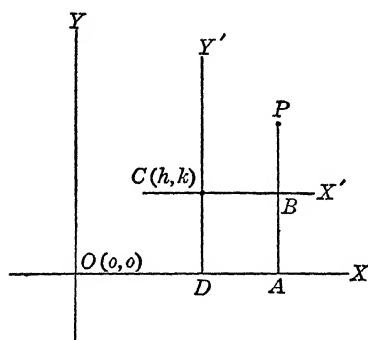


Figure 79.

We have:

$$OA = OD + DA,$$

or:

$$x = h + x'.$$

Also:

$$AP = AB + BP,$$

or:

$$y = k + y'.$$

Hence we have the *equations of translation*:

$$\boxed{\begin{array}{l} x = x' + h \\ y = y' + k \end{array}}$$

It is quite evident, then, if we are given the equation of a curve referred to any set of axes, that the equation of the same curve referred to new axes—parallel to the given axes and with origin at  $(h, k)$ —is obtained by replacing  $x$  with  $x' + h$  and  $y$  with  $y' + k$ .

### Example 1

Find the equation of the circle

$$x^2 + y^2 - 6x + 2y - 6 = 0$$

referred to parallel axes through (3, -1).

$$\begin{aligned}\text{Since} \quad & x = x' + 3 \\ \text{and} \quad & y = y' - 1,\end{aligned}$$

we substitute in the circle equation and obtain:

$$x'^2 + 6x' + 9 + y'^2 - 2y' + 1 - 6x' - 18 + 2y' - 2 - 6 = 0.$$

After simplifying this result, we have:

$$x'^2 + y'^2 = 16.$$

### *Example 2*

Remove the first degree terms from the following:

$$x^2 + y^2 - 2x + 4y - 4 = 0.$$

$$\begin{aligned}\text{Since} \quad & x = x' + h \\ \text{and} \quad & y = y' + k,\end{aligned}$$

then, substituting in the above equation, we obtain:

$$\begin{aligned}x'^2 + 2hx' + h^2 + y'^2 + 2ky' + k^2 \\ - 2x' - 2h + 4y' + 4k - 4 = 0.\end{aligned}$$

(NOTE: In order for the first degree terms to vanish, their coefficients must equal zero.)

Collecting the first degree terms, we have:

$$\begin{aligned}& x'(2h - 2), \\ & y'(2k + 4). \\ \text{Hence:} \quad & 2h - 2 = 0, \\ & 2k + 4 = 0; \\ \text{or:} \quad & h = 1, \\ & k = -2.\end{aligned}$$

Substituting again, we have:

$$\begin{aligned}x'^2 + y'^2 + 1 + 4 - 2 - 8 - 4 = 0, \\ \text{or, finally,} \quad x'^2 + y'^2 = 9.\end{aligned}$$

We might have solved Example 2 by another procedure, indicated in the succeeding text:

*Example 3*

Remove the first degree terms from the following:

$$x^2 + y^2 - 2x + 4y - 4 = 0.$$

Completing the square, we have:

$$(x - 1)^2 + (y + 2)^2 = 9.$$

Since

$$x - 1 = x'$$

and

$$y + 2 = y',$$

or

$$x = x' + 1$$

and

$$y = y' - 2,$$

hence, as before,

$$h = 1,$$

$$k = -2.$$

**Problems**

1. Find the coördinates of the points (1, 3), (-2, 5), and (3, -2) referred to parallel axes through (1, -3).

2. Find the coördinates of the points (-1, 2), (3, -2), and (x, y) referred to parallel axes through (3, -2).

3. Find the equation of the line  $x - 2y - 6 = 0$  referred to parallel axes through (2, -2).

4. Find the equation of the curve  $x^2 + y^2 - 4x + 6y - 12 = 0$  referred to parallel axes through (2, -3).

5. Find the equation of the curve  $x^2 + 2y^2 + 2x - 12y + 17 = 0$  referred to parallel axes through (-1, 3).

6. Find the equation of the curve  $3x^2 - 4y^2 - 6x - 16y - 25 = 0$  referred to parallel axes through (1, -2).

7. By translation of axes, remove the terms of first degree from:  $x^2 + y^2 - 2x + 4y - 3 = 0$ .

8. By translation of axes, remove the constant term and the term in  $x$  from:  $x^2 + 4x - 8y + 12 = 0$ .

9. Find values of  $h$  and  $k$  that will remove the constant term from:  $2x - 3y - 4 = 0$ . Are the values obtained unique?

10. By translation of axes, derive the following equations:

$$(a) \frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = :$$

$$(b) (x - \alpha)^2 = 4a(y - \beta).$$

$$(c) \frac{(x - \alpha)^2}{a^2} - \frac{(y - \beta)^2}{b^2} =$$

**78. Rotation of axes.** We now wish to change the directions of the axes without changing the origin. We do this by a rotation of the axes through an angle  $\phi$ .

Let  $OX$  and  $OY$  be an original set of coördinate axes; and  $OX'$  and  $OY'$ , a new set, with  $\phi$  the angle through which the original axes must be rotated, about the common origin  $O$ , to coincide with the new axes (Figure 80). Let  $P$  be a point whose coördinates referred to the original axes are  $x = OA$  and  $y = AP$ , and whose coördinates referred to the new axes are  $x' = OB$  and  $y' = BP$ . Drop perpendiculars from  $B$  to  $AP$  and  $OX$ .

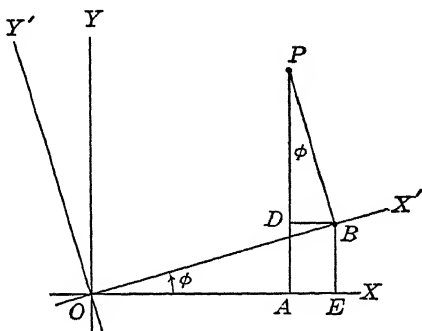


Figure 80.

Then:

$$\begin{aligned}
 x &= OA \\
 &= OE - AE \\
 &= OE - DB \\
 &= OB \cos \phi - BP \sin \phi \\
 &= x' \cos \phi - y' \sin \phi.
 \end{aligned}$$

Similarly:

$$\begin{aligned}
 y &= AP \\
 &= AD + DP \\
 &= EB + DP \\
 &= OB \sin \phi + BP \cos \phi \\
 &= x' \sin \phi + y' \cos \phi.
 \end{aligned}$$

Hence we have the *equations of rotation*:

$  \begin{aligned}  x &= x' \cos \phi - y' \sin \phi \\  y &= x' \sin \phi + y' \cos \phi  \end{aligned}  $
---

*Example 1*

By a rotation of axes, remove the term in  $y$  from:  $3x + 4y - 10 = 0$ .

Substituting equations of rotation, we have:

$$3(x' \cos \phi - y' \sin \phi) + 4(x' \sin \phi + y' \cos \phi) - 10 = 0.$$

Or:

$$x'(3 \cos \phi + 4 \sin \phi) + y'(4 \cos \phi - 3 \sin \phi) - 10 = 0.$$

(NOTE: For the term in  $y'$  to vanish, its coefficient must equal zero.) Hence:

$$4 \cos \phi - 3 \sin \phi = 0,$$

$$\text{or:} \quad \frac{\sin \phi}{\cos \phi} = \frac{4}{3}.$$

$$\text{Then:} \quad \tan \phi = \frac{4}{3}.$$

$$\text{Therefore:} \quad \sin \phi = \frac{4}{5},$$

$$\cos \phi = \frac{3}{5}.$$

Hence, substituting further, we obtain:

$$x' \left( \frac{9}{5} + \frac{16}{5} \right) + y' \left( \frac{12}{5} - \frac{12}{5} \right) - 10 = 0,$$

$$\text{or:} \quad 5x' - 10 = 0,$$

$$\text{or:} \quad x' = 2.$$

**Problems**

1. By a rotation of axes, remove the term in  $x$  from:  $3x - 4y - 6 = 0$ .

2. By a rotation of axes, remove the term in  $y$  from:  $5x + 12y - 7 = 0$ .

3. By a rotation of axes, remove the term in  $y$  from:  $x^2 + y^2 - 2x - 2y = 0$ .

4. After the axes are rotated through  $30^\circ$ , find the equation of the line  $3x - 2y + 6 = 0$ .

5. After the axes are rotated through  $45^\circ$ , find the equation of the line  $3x + 3y - 10 = 0$ .

6. After the axes are rotated through  $90^\circ$ , find the equation of the circle  $x^2 + y^2 = 25$ .

7. After the axes are rotated through  $45^\circ$ , find the equation of the circle  $x^2 + y^2 = 25$ .

8. After the axes are rotated through  $45^\circ$ , find the equation of the curve  $xy = 6$ .

9. After the axes are rotated through  $45^\circ$ , find the equation of the curve  $xy = 4$ .

10. Remove the  $xy$  term from:  $xy = 10$ .

**79. Removal of the  $xy$  term.** In the four preceding chapters, we discussed the various types of second degree equations containing no  $xy$  terms, and the curves arising from these types. We propose to show here that there is a transformation which will always remove the  $xy$  term from the most general equation of the second degree; for example:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Thus, having reduced the equation to one with which we are familiar, the procedure enables us to handle completely the general second degree equation. We now proceed to the finding of this transformation.

We are given

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Substituting in the equation the following values:

$$x = x' \cos \phi - y' \sin \phi,$$

$$y = x' \sin \phi + y' \cos \phi,$$

we have, after collecting terms,

$$\begin{aligned} & x'^2(A \cos^2 \phi + B \sin \phi \cos \phi + C \sin^2 \phi) \\ & + x'y'(B \cos^2 \phi - B \sin^2 \phi + 2C \sin \phi \cos \phi - 2A \sin \phi \cos \phi) \\ & + y'^2(A \sin^2 \phi - B \sin \phi \cos \phi + C \cos^2 \phi) \\ & + x'(D \cos \phi + E \sin \phi) \\ & + y'(E \cos \phi - D \sin \phi) \\ & + F = 0. \end{aligned}$$

(NOTE: In order for the  $x'y'$  term to vanish, its coefficient must equal zero.) Hence:

$$B(\cos^2 \phi - \sin^2 \phi) - (A - C)2 \sin \phi \cos \phi = 0.$$

But, from Section 35,

$$\cos^2 \phi - \sin^2 \phi = \cos 2\phi,$$

and

$$2 \sin \phi \cos \phi = \sin 2\phi.$$

Therefore:

$$B \cos 2\phi = (A - C) \sin 2\phi,$$

or:

$$\frac{\sin 2\phi}{\cos 2\phi} = \frac{B}{A - C}.$$

Hence we have the following *transformation*

$$\boxed{\tan 2\phi = \frac{B}{A - C}}$$

In problems generally, we need the values for  $\sin \phi$  and  $\cos \phi$ , in order to substitute them in the rotation formula. Such values are easily found from a variation of the half-angle formulas (Section 36); that is:

$$\begin{aligned}\sin \phi &= \sqrt{\frac{1 - \cos 2\phi}{2}}, \\ \cos \phi &= \sqrt{\frac{1 + \cos 2\phi}{2}}.\end{aligned}$$

Before proceeding to an example, let us make a few observations concerning the above material. In the first place, the translation and the rotation formulas of Section 78 are of the first degree; hence, when they are substituted in an equation, the degree of the equation is certainly not raised. Moreover, the degree is not lowered; for, if it were, then substituting further to restore the original axes would yield

an equation of lower degree than the original degree. Thus the degree of an equation remains unchanged by a transformation of coördinates.

Therefore it follows that, by the transformation

$$\tan 2\phi = \frac{B}{A - C},$$

the general equation of the second degree

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

may be reduced to the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0,$$

where  $A'$  and  $C'$  cannot *both* be zero.

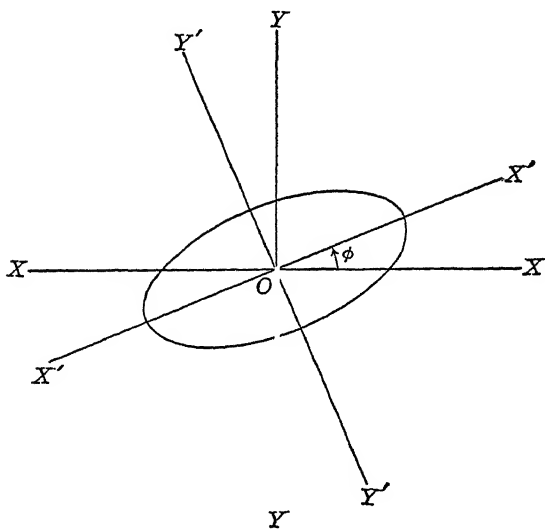


Figure 81.

However, we have previously shown that this equation always represents some type of conic, including degenerate, imaginary, or point conics. Hence we say: *Every second degree equation represents a conic, and every conic is represented by an equation of the second degree.*

A typical situation is illustrated by the ellipse in Figure 81.

In Section 80, we shall show how the type of conic may be determined by certain relations among the coefficients  $A$ ,  $B$ , and  $C$ .

*Example 1*

Remove the  $xy$  term from:  $5x^2 - 4xy + 2y^2 = 6$ .

Here,  $A = 5$ ,  $B = -4$ , and  $C = 2$ .

$$\text{Hence:} \quad \tan 2\phi = \frac{-4}{5-2} = -\frac{4}{3}.$$

$$\text{Therefore:} \quad \sin 2\phi = \frac{4}{5},$$

$$\cos 2\phi = -\frac{3}{5}.$$

(Since we assume  $\phi$  to be acute in this example,  $2\phi$  is thus in the second quadrant.) Then:

$$\sin \phi = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \sqrt{\frac{8}{10}} = \sqrt{\frac{4}{5}},$$

$$\cos \phi = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \sqrt{\frac{2}{10}} = \sqrt{\frac{1}{5}}.$$

Substituting in the original equation, we have:

$$\begin{aligned} 5\left(x'\sqrt{\frac{1}{5}} - y'\sqrt{\frac{4}{5}}\right)^2 - 4\left(x'\sqrt{\frac{1}{5}} - y'\sqrt{\frac{4}{5}}\right)\left(x'\sqrt{\frac{4}{5}} + y'\sqrt{\frac{1}{5}}\right) \\ + 2\left(x'\sqrt{\frac{4}{5}} + y'\sqrt{\frac{1}{5}}\right)^2 \end{aligned}$$

$$\text{or:} \quad x'^2 + 6y'^2 = 6.$$

This equation is in the form of an ellipse.

*Example 2*

Remove the  $xy$  term from:  $xy = k$ .

Here,  $A = 0$ ,  $B = 1$ , and  $C = 0$ .

$$\text{Hence:} \quad \tan 2\phi = \frac{1}{0} = \infty.$$

$$\text{Therefore:} \quad 2\phi = 90^\circ,$$

$$\text{or:} \quad \phi = 45^\circ.$$

Substituting in the original equation, we have:

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) = k,$$

or: 
$$\frac{1}{2}(x' - y')(x' + y') = k,$$

or: 
$$x'^2 - y'^2 = 2k.$$

This equation is in the form of an equilateral hyperbola with the coördinate axes as asymptotes.

### Problems

Remove the  $xy$  terms, and determine the type of curve in each of the following. Given:

1.  $xy = 4.$
2.  $2xy = -7.$
3.  $x^2 + 3xy - 3y^2 - 4 = 0.$
4.  $5x^2 - 6xy + 5y^2 - 8 = 0.$
5.  $x^2 + 4xy + y^2 = 2.$
6.  $3x^2 - 3xy - y^2 = 10.$
7.  $x^2 + xy - 5x - 3y + 6 = 0.$
8.  $x^2 - 4xy + 4y^2 - 4x - 2y + 8 = 0.$
9.  $x^2 - 2xy + 2y^2 - 2x = 0.$
10.  $y^2 + xy - 2x^2 - 4 = 0.$
11.  $y^2 - 2xy + 2x = 0.$
12.  $8x^2 + 12xy + 17y^2 - 20 = 0.$
13.  $x^2 + 2xy + y^2 + 2x + 6y = 0.$
14.  $3x^2 + 4xy + 6x + 4y - 1 = 0.$
15.  $x^2 + 24xy - 6y^2 - 30 = 0.$

**80. Invariants; classifications of types of conics.** In this section we shall find how to determine at a glance, by certain relations among the three coefficients  $A$ ,  $B$ , and  $C$ , the type of conic represented by the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Consider the above equation in connection with the rotation formulas:

$$\begin{aligned}x &= x' \cos \phi - y' \sin \phi, \\y &= x' \sin \phi + y' \cos \phi.\end{aligned}$$

Substituting in the original equation, we obtain:

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0,$$

where, as determined in Section 79,

$$(1) \quad A' = A \cos^2 \phi + B \sin \phi \cos \phi + C \sin^2 \phi$$

$$(2) \quad B' = B \cos 2\phi - (A - C) \sin 2\phi$$

$$(3) \quad C' = A \sin^2 \phi - B \sin \phi \cos \phi + C \cos^2 \phi$$

and

$$D' = D \cos \phi + E \sin \phi$$

$$E' = E \cos \phi - D \sin \phi$$

$$F' = F$$

(These last three relations are not necessary in our particular problem.)

We shall first find some interesting relations that exist between  $A$ ,  $B$ ,  $C$  and  $A'$ ,  $B'$ ,  $C'$ .

Adding equations (1) and (3), we obtain:

$$A' + C' = A(\cos^2 \phi + \sin^2 \phi) + C(\cos^2 \phi + \sin^2 \phi),$$

or:

$$(4) \quad A' + C' = A + C.$$

Observe that the relation between the primed quantities,  $A' + C'$ , is equal to the same relation between the unprimed quantities,  $A + C$ . We call  $A + C$  an *invariant*.

We shall now derive two other invariants. Subtracting (3) from (1), we have:

$$A' - C' = (A - C)(\cos^2 \phi - \sin^2 \phi) + 2B \sin \phi \cos \phi,$$

or:

$$(5) \quad A' - C' = (A - C) \cos 2\phi + B \sin 2\phi.$$

Observe that  $(A - C)$  is *not* an invariant. But if we square (5), add  $B'^2$  to the left-hand side of the equation, and then add the value of  $B'^2$  from (2) to the right side of the equation, we obtain:

$$(A' - C')^2 + B'^2 =$$

$$(A - C)^2 \cos^2 2\phi + 2B(A - C) \sin 2\phi \cos 2\phi + B^2 \sin^2 2\phi \\ + (A - C)^2 \sin^2 2\phi - 2B(A - C) \sin 2\phi \cos 2\phi + B^2 \cos^2 2\phi,$$

or:

$$(6) (A' - C')^2 + B'^2 = (A - C)^2 + B^2.$$

Thus we have the invariant:  $(A - C)^2 + B^2$ .

Finally, square (4), and then subtract from (6). The resulting equation is

$$\begin{aligned} A'^2 - 2A'C' + C'^2 + B'^2 - A'^2 - 2A'C' - C'^2 \\ = A^2 - 2AC + C^2 + B^2 - A^2 - 2AC - C^2, \end{aligned}$$

or:

$$(7) B'^2 - 4A'C' = B^2 - 4AC.$$

Hence  $B^2 - 4AC$  is an invariant.

Moreover, it is the invariant  $B^2 - 4AC$  that will determine the various types of conics. Consider the transformation

$$\tan 2\phi = \frac{B}{A - C}$$

applied to the general second degree equation. We know, from Section 79, that  $B' = 0$ . Hence, the resulting equation of the second degree is:

$$(8) A'x'^2 + D'x'y' + E'y'^2 + F' = 0;$$

and (7) becomes:

$$(9) B^2 - 4AC = -4A'C'.$$

Now, if either  $A'$  or  $C'$  is zero, (8) represents a parabola. However, from (9), we know that  $B^2 - 4AC = 0$ . Again, if  $A'$  and  $C'$  have the same sign, (8) represents an ellipse. However, from (9), we know that  $B^2 - 4AC$  is negative. Finally, if  $A'$  and  $C'$  have opposite signs, (8) represents a hyperbola. However, from (9), we know that  $B^2 - 4AC$  is positive. The process is also reversible.

Hence we have:

$$\text{Parabola: } B^2 - 4AC = 0$$

$$\text{Ellipse: } B^2 - 4AC < 0$$

$$\text{Hyperbola: } B^2 - 4AC > 0$$

In each of the above formulas, it should be understood that degenerate and imaginary cases are included.

*Example*

Classify:  $3x^2 - 4xy - 2y^2 + x - y - 3 = 0$ .

Here,  $A = 3$ ,  $B = -4$ , and  $C = -2$ .

$$\begin{aligned}\text{Hence:} \quad B^2 - 4AC &= 16 - 4(3)(-2) \\ &= 16 + 24 \\ &= 40.\end{aligned}$$

Therefore the form is an hyperbola.

**Problems**

Classify:

1.  $3x^2 - 2xy + 4y^2 - 7x + 3y - 10 = 0$ .
2.  $x^2 + xy - y^2 + x - y - 6 = 0$ .
3.  $2x^2 + 4xy + 2y^2 = 9$ .
4.  $x^2 + 3xy - y^2 + 2x - y - 4 = 0$ .
5.  $2x^2 + 4xy + y^2 - 5 = 0$ .
6.  $3xy - 2x + y - 6 = 0$ .
7.  $3x^2 + 3y^2 - x - 2y - 4 = 0$ .
8.  $(x + 2y)^2 = 4x$ .
9.  $(x + 2y)^2 = 4$ .
10.  $xy + 3y^2 - 2x + y - 3 = 0$ .

## INDEX



# INDEX

[REFERENCES ARE TO PAGE NUMBERS]

- Abscissa, 36
- Addition theorems, 76
- Ambiguous case, 54
- Angle:
  - between two lines, 131
  - bisectors, 156
  - definition of, 21
  - functions of, 21, 22, 38
  - negative, 35
  - of depression, 31
  - of elevation, 30
  - positive, 35
- Anti-logarithm, 12
- Asymptote, 199
- Axis:
  - conjugate, 196
  - major, 183
  - minor, 184
  - of coordinates, 37
  - of parabola, 173
  - radical, 169
  - transverse, 196
- Base:
  - of a logarithm, 7
  - of a number, 3
- Briggs logarithms, 10
- Center:
  - of circle, 162
  - of ellipse, 184
  - of hyperbola, 197
  - radical, 169
- Characteristic, 11
- Chord—common of two circles, 169
- Circle, 162
  - general equation of, 164
  - standard equation of, 162
  - through intersection of two circles, 168
  - through three points, 165
  - unit, 43, 44
- Circular measure, 100
- Common logarithms, 10
- Compound interest, 18
- Conics, 172
  - definition of, 172
  - determination of types of, 221
  - types of, 172
- Conjugate axis of hyperbola, 196
- Conjugate hyperbola, 207
- Coordinate axes, 37
  - rotation of, 209, 213
  - translation of, 209, 210
- Coordinates, 37
  - transformation of, 209
- Cosecant:
  - definition of, 22, 38
  - variation of, 44
- Cosine:
  - definition of, 21, 38
  - variation of, 43
- Cosines:
  - law of, 57
- Cotangent:
  - definition of, 22, 38
  - variation of, 44
- Curve, 136
  - condition that a point lie on, 136
- Depression, angle of, 31
- Directed distance, 35
- Directrix, 172
  - of ellipse, 181
  - of hyperbola, 194
  - of parabola, 172
- Distance:
  - between two parallel lines, 151
  - between two points, 123
  - from a line to a point, 153
  - of a line from the origin, 148
- Division of line segment, 126
- Double angle formulas, 79, 80
- Eccentricity, 172
  - of ellipse, 181
  - of hyperbola, 194
  - of parabola, 172
- Elevation, angle of, 30

- Ellipse, 181  
   center of, 184  
   criterion for, 221  
   definitions of, 172, 181, 193  
   general equation of, 191  
   imaginary, 191  
   major axis of, 183  
   minor axis of, 184  
   point, 191  
   property of, 190  
   standard equation of, 183, 185, 190  
   vertices of, 181  
 Equation:  
   first degree, 144  
   normal, 149  
   second degree, 215, 219  
   trigonometric, 28, 50  
 Equilateral hyperbola, 208  
 Exponents, 3  
  
 Focus, 172  
   of ellipse, 181  
   of hyperbola, 194  
   of parabola, 172  
 Fractional exponents, 5  
 Functions:  
   trigonometric, 21, 22, 38  
 Functions of:  
    $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , 23  
    $90^\circ$ ,  $180^\circ$ ,  $270^\circ$ ,  $360^\circ$ , 40-42  
    $90^\circ - \theta$ , 25  
    $90^\circ + \theta$ , 72  
    $n90^\circ \pm \theta$ , 72  
    $180^\circ - \theta$ , 46  
    $n180^\circ \pm \theta$ , 48  
    $-\theta$ , 49  
    $\alpha + \beta$ , 76  
    $\alpha - \beta$ , 78  
    $2\alpha$ , 79  
    $\frac{\alpha}{2}$ , 82, 83  
 Fundamental identities, 66, 67  
  
 General equation:  
   of circle, 164  
   of ellipse, 191  
   of first degree, 144  
   of hyperbola, 204  
   of parabola, 178  
   of second degree, 215, 219  
   of straight line, 144  
  
 Half-angle formulas, 82, 83  
 Hyperbola, 194  
   asymptotes of, 199  
   center of, 197  
   conjugate, 207  
   conjugate axis of, 196  
   criterion for, 221  
   definitions of, 172, 194, 207  
   degenerate, 204  
   equilateral, 208  
   general equation of, 204  
   property of, 202  
   standard equation of, 195, 197, 203, 204  
   transverse axis of, 196  
   vertices of, 194  
  
 Identities, 66  
   fundamental, 66, 67  
 Imaginary:  
   circle, 165  
   ellipse, 191  
   number, 165, 169  
   point, 169  
 Infinity, 41  
 Initial side, 21  
 Inscribed circle, 168  
   radius of, 98  
 Intercept form of straight line, 144  
 Intercepts on the axes, 142  
 Interpolation, 13  
 Intersections of two curves, 137  
 Intrinsic property:  
   of ellipse, 189  
   of hyperbola, 202  
   of parabola, 176  
 Invariant, 220  
  
 Latus rectum:  
   of ellipse, 184  
   of hyperbola, 197  
   of parabola, 175  
 Law of cosines, 57  
 Law of sines, 51  
 Law of tangents, 93  
 Laws of exponents, 3, 4, 6  
 Laws of logarithms, 8, 9, 10  
 Locus, 136  
   equation of, 136  
 Logarithms, 7  
   base of, 8  
   Briggs, 10  
   common, 10

- Logarithms (*cont.*):  
 laws of, 8, 9, 10  
 of functions, 112–116  
 of numbers, 108, 109  
 to base  $e$ , 10  
 use of, 12, 15
- Mantissa, 11
- Negative angles, 35  
 Negative direction, 36  
 Normal, 149  
 Normal form, 149
- Oblique triangle, 51  
 Ordinate, 37  
 Origin of coordinates, 37
- Parabola, 172  
 axis of, 173  
 criterion for, 221  
 degenerate, 179  
 general equation of, 178  
 property of, 176  
 standard equation of, 173, 174, 177  
 vertex of, 172
- Parallel lines, 130, 145, 151  
 Perpendicular lines, 130, 146  
 Principal angle, 73  
 Projection, 73  
 theorems on, 73, 74
- Property:  
 of ellipse, 189  
 of hyperbola, 202  
 of parabola, 176
- Pythagorean law, 22
- Quadrants, 37
- $r$  formulas, 95  
 Radian, 100  
 Radical axis, 169  
 Radical center, 169  
 Radius:  
 of circle, 162  
 of circumscribed circle, 62  
 of inscribed circle, 98  
 vector, 37
- Ratio formula, 127  
 Removal of  $xy$  term, 215, 216  
 Right triangle, solution of, 29  
 Roots, square, tables, 118, 119
- Rotation of axes, 209, 213  
 formulas for, 213
- Secant:  
 definition of, 22, 38  
 variation of, 44
- Second directrix:  
 of ellipse, 187  
 of hyperbola, 199
- Second focus:  
 of ellipse, 187  
 of hyperbola, 199
- Signs of the functions, 39
- Sine:  
 definition of, 21, 38  
 variation of, 43
- Sines, law of, 51
- Slope:  
 definition of, 129  
 of a line through two points, 130  
 of two parallel lines, 130  
 of two perpendicular lines, 131  
 point-slope form, 141  
 slope-intercept form, 143
- Solution of triangles, 29, 51
- Straight line, 140  
 general equation, 144  
 intercept form, 144  
 normal form, 149  
 parallel to an axis, 140  
 point-slope form, 141  
 slope-intercept form, 143  
 through the intersection of two lines, 157  
 two point form, 143
- Tables:  
 of logarithms of numbers, 108, 109  
 of logarithms of trigonometric functions, 112–116  
 of squares and square roots, 118, 119  
 of trigonometric functions, 112–116
- Tangent:  
 definition of, 21, 38  
 of a half-angle in terms of the sides of a triangle, 94  
 of the angle between two lines, 132  
 variation of, 43
- Tangents, law of, 93
- Terminal side, 21
- Transformation of coordinates, 209

Translation of axes, 209, 210  
Transverse axis of hyperbola, 196  
Triangle:  
  area of, 62, 96, 99  
  of reference, 38  
  oblique, 51  
  right, 29  
  solution of, 29, 51  
Trigonometric functions, 21, 38

Unit circle, 43, 44

Variation of the trigonometric functions, 42

Vector, radius, 37

Vertex:  
  of ellipse, 181  
  of hyperbola, 194  
  of parabola, 172